Abstract of thesis entitled

ON ASAI'S FUNCTION ANALOGOUS TO $\log|\eta(z)|$

submitted by

Cangxiong Chen

for the degree of Doctor of Philosophy
at the University of Cambridge
in December 2015

Kronecker's first limit formula describes the constant term in the Laurent expansion of a non-holomorphic Eisenstein series at one of its poles. Asai generalised the limit formula to Eisenstein series of level one defined for a number field with class number one and obtained a function analogous to the logarithm of the absolute value of the eta function. In this thesis we reformulate Asai's function adelically using the theory of admissible representations for GL_2 and simultaneously remove the restriction on class number and level. As an application of the method, we give explicit computations of the Rankin-Selberg integral with two Eisenstein series and a cusp form.

ON ASAI'S FUNCTION ANALOGOUS TO $\log|\eta(z)|$

Cangxiong Chen

Hughes Hall

Department of Pure Mathematics and Mathematical Statistics
University of Cambridge

This dissertation is submitted for the degree of Doctor of Philosophy
at the University of Cambridge.

December 2015

Declarations

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Introduction and specified in the text. It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Introduction and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University of similar institution except as declared in the Introduction and specified in the text.

Cangxiong Chen

Acknowledgements

I would like to express my sincere gratitude to Prof. Tony Scholl for his suggestion of the topic and his continuous guidance and support throughout my PhD study. I feel very grateful for Dr Teruyoshi Yoshida for his valuable advice and mental support. I would like to thank Dr Tobias Berger, Dr Tom Fisher and Prof. John Coates for their comments. I would like to extend my thanks to Jack Lamplugh, Chris Blake and Emile Bouaziz for many helpful discussions.

I am indebted to my parents, Mr Zhengzhao Chen and Ms Yiping Sun. Without their generous financial support, my study in Cambridge would not have been possible. I would like to thank the department and Hughes Hall for their support throughout my PhD. I would like to thank Ms Sarah Dodd who provided strong administrative support in the last stage of my writing-up.

Part of the writing-up of this thesis was supported by a studentship from the Cambridge Philosophical Society.

To my parents Mr Zhengzhao Chen and Ms Yiping Sun

Contents

Introduction

The aim of this thesis is to generalise Asai's function to arbitrary number fields, and to automorphic forms of higher levels. While Asai worked classically, we use the theory of adelic Eisenstein series developed mainly by Jacquet and Langlands in [JL70] and Jacquet in [Jac72]. Let F be a number field with class number one. In [Asa70], Asai considered an Eisenstein series which is an analog of the non-holomorphic Eisenstein series defined on the upper half plane. From its Fourier expansion Asai obtained a function h which is annihilated by the Laplace-Beltrami operator. It has a modular transformation property with respect to $\mathrm{SL}_2(\mathcal{O}_F)$ and is associated to some Dirichlet series by the Mellin transform. Section 1.1 introduces the classical limit formula considered by Kronecker and gives a historical account on the work in generalising the limit formula. Section 1.2 reviews Asai's original constructions. Section 1.3.1 and section 1.3.2 give a simple characterisation of Asai's function and an alternative KLF, which describes the first order derivative of Eisenstein series at zero of the auxiliary variable and the proof does not depend on the computation of the Fourier expansion of Eisenstein series. Another consideration in a similar spirit is Colin de Verdière's work on pseudo-Laplacians in [CdV83]. In appendix 1.A.1, we review part of his work in [CdV83] on characterising Eisenstein series using certain extension of the Laplacian operator on the complex upper half plane and properties of its resolvent. He showed that analytic continuations of Eisenstein series can be obtained as a result of this characterisation. Then in 1.A.2, we show how we can generalise CdV's method to characterise Eisenstein series with weights and obtain analytic continuations up to the critical line $\mathrm{Re}\, s = \frac{1}{2}$.

In chapter two we introduce adelic Eisenstein series on GL_2 based on the theory of admissible representations. The idea goes back to [Tat67] and the constructions are contained in [JL70] and [Jac72]. The main idea is to construct certain 'nice' sections in the admissible representation using certain test functions called Schwartz functions. These functions behave nicely under the Fourier transform and the analytic properties of the Eisenstein series obtained in this way can be easily deduced. Furthermore, by choosing Schwartz functions properly we could construct with ease Eisenstein series which are invariant under given open compact subgroups of $\mathrm{GL}_2(\mathbb{A}^\infty)$ (i.e. Eisenstein series with nontrivial levels), with $\mathbb{A}^\infty$ being the finite adeles of the number field F. In sections 2.1 and 2.2 we review some basic notions and properties of automorphic forms on GL_2 and admissible representations of GL_2. Section 2.3 gives a summary of the notions in harmonic analysis on locally compact groups that will be used repeatedly in the computations thereafter. Section 2.4 summarises Jacquet's construction of Eisenstein series in [Jac72]. Various computations of the section f and its image $\mathcal{M}f$ under the intertwining operator $\mathcal{M}$ are given and we recover a few classical Eisenstein series by specialisation. In the end, we answer the question of which sections f can be constructed using Schwartz functions.

In sections 3.1 and 3.2 of chapter three we summarise facts about Whittaker and Kirillov models associated to an infinite dimensional irreducible admissible representation. In section 3.3 we give computations of local Whittaker functions in the following cases:

(i) archimedean places,

(ii) nonarchimedean places where the characters are unramified,

(iii) nonarchimedean places where the characters are ramified.

Chapter four presents results of generalised Kronecker limit formulae for adelic Eisenstein series. In [Jac72] Jacquet proved the analytic continuation and functional equation by applying the Poisson summation formula to the defining formula for the Eisenstein series. However, we choose to review a method which is based on an analysis of the intertwining operator appeared in [GS88, Chap 1] and [Bum97, Sec 3.7] and applies to more general Eisenstein series. This is done in section 4.1. As a consequence, it can be shown that the Eisenstein series has at worst two simple poles. With that knowledge in hand, we then give different limit formulae according to ramifications of the characters defining the Eisenstein series. The computations can be classified into two cases:

(1) every nonarchimedean place is unramified.

(2) a finite nonempty set of archimedean places ramifies. In this case the Eisenstein series has no pole.

These limit fomulae are obtained from Fourier coefficients given in Chapter three. Analogs of Asai's function are obtained as by-products of the generalised limit formulae.

Chapter five applies the constructions of adelic Eisenstein series and computations of their Fourier expansions to the study of Rankin-Selberg integrals. We know from [Jac72] that a Rankin-Selberg integral represents certain L-function attached to the product of two admissible representations of GL_2. As a result, we can obtain analytic continuation and functional equation of the corresponding L-function. However, there does not seem to be explicit results regarding the computation of the integral itself besides cases at good primes. In this chapter, we consider Rankin-Selberg integral of two adelic Eisenstein series and a cusp form. We give results in various local cases including the ramified case as well as the archimedean case. At the end of section 5.1, we obtain results of the global Rankin-Selberg integral, including a special case when the base field is totally real, the cusp form and one of the Eisenstein series are both holomorphic. Our results can be regarded as generalisations of Scholl's work in sections 4.4 to 4.5 of [Sch98].

Chapter 1

Asai's function analogous to $\log|\eta(z)|$

In this chapter we give a brief historical review of Kronecker's first limit formula. In section 1.1, we start with the limit formula for zeta functions attached to absolute ideal classes of an imaginary quadratic field, very much what Kronecker himself considered. Then we briefly mention some work being done in generalising the limit formula to other number fields. In section 1.2, we summarise T.Asai's work in the paper [Asa70].

1.1 Kronecker's first limit formula

Let $z = x + iy, x, y \in \mathbb{R}, y > 0$. Define a quadratic form $Q(u, v) = y^{-1}(u + vz)(u + v\bar{z})$. The zeta function associated to $Q(u, v)$ is defined as:

$$\zeta_Q(s) = \sum_{(m,n)\neq(0,0)} Q(m, n)^{-s}, \quad \operatorname{Re} s > 1. \tag{1.1.1}$$

This zeta function can be analytically continued to the whole s-plane with a simple pole at $s = 1$.

Kronecker's first limit formula (written 'KLF' for short) describes the constant term in the Laurent expansion:

$$\lim_{s\to 1}\left\{\zeta_Q(s)-\frac{\pi}{s-1}\right\}=2\pi(C-\log 2-\frac{1}{2}\log y-2\log|\eta(z)|) \tag{1.1.2}$$

where C is the Euler-Mascheroni constant and $\eta(z)$ is the Dedekind eta function:

$$\eta(z)=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n),\quad q=e^{2\pi iz}. \tag{1.1.3}$$

This limit formula has many applications in number theory.

For example, the zeta function attached to an ideal class A of a number field F is defined as follows:

$$\zeta(s,A)=\sum_{\mathfrak{a}\in A}\frac{1}{\mathrm{N}(\mathfrak{a})^s},\quad \mathrm{Re}\, s>1. \tag{1.1.4}$$

The analytic class number formula implies that $\zeta(s,A)$ has analytic continuation to the whole s-plane, and we can write:

$$\zeta(s,A)=\frac{R}{s-1}+c(A)+O(s-1), \tag{1.1.5}$$

where the residue R is independent of the ideal class A.

The constant term $c(A)$ is dependent on A and appears in the L-series attached to an ideal class character χ. More precisely, if χ is a nontrivial character of the ideal class group, then

$$\begin{aligned} L(s,\chi) &= \sum_{\mathfrak{a}\subset\mathcal{O}}\chi(\mathfrak{a})\,\mathrm{N}(\mathfrak{a})^{-s} \\ &= \sum_{A}\chi(A)\zeta(s,A) \\ &= \sum_{A}\chi(A)\left\{\frac{R}{s-1}+c(A)+O(s-1)\right\} \\ &= \sum_{A}\chi(A)c(A)+O(s-1), \end{aligned} \tag{1.1.6}$$

by character orthogonality.

As a result, $L(1,\chi) = \sum_A \chi(A)c(A)$. In the case when F is imaginary quadratic, this gives an explicit formula for $L(1,\chi)$ via the KLF.

Let $d < 0$ be the discriminant of F. Let A be an ideal class. Fix $\mathfrak{b} \in A^{-1}$. It defines a map from A to principal ideals contained in $\mathfrak{b} : \mathfrak{a} \mapsto \mathfrak{b}\mathfrak{a} = (\alpha)$.

Write

$$\begin{aligned} \zeta(s,A) &= N(\mathfrak{b})^s \sum_{0 \neq (\alpha) \subset \mathfrak{b}} \frac{1}{N(\alpha)^s} \\ &= \frac{1}{|\mathcal{O}_F^\times|} N(\mathfrak{b})^s \sum_{0 \neq \alpha \in \mathfrak{b}} \frac{1}{N(\alpha)^s}. \end{aligned} \tag{1.1.7}$$

Assume $\mathfrak{b}$ is generated by $1, z$ as a $\mathbb{Z}$-module with $\operatorname{Im} z > 0$. We have $N(\mathfrak{b}) = \frac{|z-\bar{z}|}{\sqrt{-d}} = \frac{2y}{\sqrt{-d}}$, with $y = \operatorname{Im} z$. Then we can write $\zeta(s,A)$ as:

$$\zeta(s,A) = \frac{1}{|\mathcal{O}_F^\times|} \cdot [4(-d)^{-1}]^{\frac{s}{2}} \sum_{(m,n) \neq 0} \frac{y^s}{|mz+n|^{2s}}, \quad \operatorname{Re} s > 1. \tag{1.1.8}$$

This gives an expression for $\zeta(s,A)$ in terms of the quadratic form $Q(u,v)$ in (1.1.1).

Combining this with the KLF, (1.1.2) gives an explicit expression for $L(1,\chi)$ in terms of values of $\log|\eta(z)|$. This was used by Stark to verify his conjectures on special L-values for imaginary quadratic fields (ref. [Sta75]). Ramachandra also made use of KLF in constructing ray class fields of imaginary quadratic fields (ref. [Ram64]). And in [Rob73], Robert developed the notion of elliptic units in constructing certains units in the Abelian extensions of imaginary quadratic fields.

Another interesting aspect of the limit formula is the study of Dedekind sums appearing in the transformation formula of $\log|\eta(z)|$. For a nice historical account of this problem, refer to [Ati87].

One may ask the question of generalising KLF to an arbitrary number field. It turns out that even for a real quadratic field, the question of finding a function analogous to $\log|\eta(z)|$ in closed form is a nontrivial one. The difficulty lies in the presence of units of infinite order in the field. Progress was made first by Hecke, who came up with a non-closed form involving an integral of $\log|\eta(z)|$. Siegel's notes ([Sie65]) provides a detailed account of Hecke's work on KLF. Zagier's paper ([Zag75]) gives a nice historical account of the progress.

Asai dealt with this task by considering arbitrary number fields of class number one and developing KLF for a certain type of non-holomorphic Eisenstein series [Asa70]. He expressed his function analogous to $\log|\eta(z)|$ using Bessel functions. Asai's function h is analogous to $\log|\eta(z)|$ in following ways: it is modular with respect to the full modular group, it is annihilated by the Laplacian-Beltrami operator of the underlying space and it is associated to Dirichlet series via Mellin transform. We will review his work in section 1.2 below.

Continuing down the same line, Jorgenson and Lang's work in [JL99] removed the restriction on class number imposed by Asai. They reduced the problem to the class number one case by considering developing limit formulae for partial Eisenstein series associated to one ideal class. They obtained a generalisation of Asai's function which is a sum over the class group of those attached to each ideal class. Each summand is expressed in terms of Bessel functions.

We believe that to generalise the Asai function, it is most convenient to work adelically. In this direction, Driencourt's work [Dri86] is closest to ours in spirit. He made use of a type of Eisenstein series formulated in Godement's paper [God64]. Besides reformulating Asai's function adelically, he also considered applying the limit formula to Eisenstein series with levels. However, his formulation of Eisenstein series is different from ours.

1.2 Asai's function

In this section, we give a brief summary of Asai's work in [Asa70]. Recall the Eisenstein series mentioned in last section,

$$E(z,s) = \sum_{(m,n)\neq 0} \frac{y^s}{|mz+n|^{2s}}, \quad \operatorname{Re} s > 1. \tag{1.2.1}$$

One can rewrite this as a sum over the coset space $\Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})$ where

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\} \subset \mathrm{SL}_2(\mathbb{Z}). \tag{1.2.2}$$

This is,

$$E^*(z,s) = \sum_{\sigma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} y(\sigma z)^s, \tag{1.2.3}$$

where $y(z) = \mathrm{Im}\, z$. Notice that $E(z,s) = 2\zeta(2s)E^*(z,s)$, where $\zeta(s)$ is the Riemann zeta function.

Based on this formulation of summing over groups instead of lattices, one can write down an Eisenstein series for an arbitrary number field F with class number one.

Let the degree of F be $n = r_1 + 2r_2$, where r_1, r_2 are the number of real and complex places respectively. Let $H = \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ where $\mathcal{H}$ is the complex upper half plane and $\mathbb{H}$ the quaternion upper half plane given by:

$$\mathbb{H} = \left\{ \begin{pmatrix} x & -y \\ y & \bar{x} \end{pmatrix} \middle| x \in \mathbb{C}, y > 0 \right\}. \tag{1.2.4}$$

For $z = (z_v) \in H$, write $Ny(z) = \prod_{v=1}^{r_1+r_2} y(z_v)^{e_v}$ where

$$e_v = \begin{cases} 1 & v \quad \text{real} \\ 2 & v \quad \text{complex.} \end{cases} \tag{1.2.5}$$

Denote by $\mathcal{O}$ the ring of integers of F and

$$\Gamma_\infty = \left\{ \begin{pmatrix} u & * \\ & u^{-1} \end{pmatrix} \middle| u \in \mathcal{O}^\times \right\} \subset \mathrm{SL}_2(\mathcal{O}). \tag{1.2.6}$$

Define the following Eisenstein series for F:

$$E^*(z,s) = \sum_{\sigma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathcal{O})} N(y(\sigma z))^s, \quad \mathrm{Re}\, s > 1. \tag{1.2.7}$$

Since the class number of F is one, the different ideal is generated by a single element, i.e. $\mathfrak{d}^{-1} = (\varpi), \varpi \in \mathcal{O}$. Denote the absolute value of the discriminant of F by D.

By using the inversion formula for Hecke's theta series, one can write the Fourier expansion of E as follows (ref. [Asa70, p.203])

$$\begin{aligned} E(z,s) &:= 2\zeta_F(2s)E^*(z,s) \\ &= Ny(z)^s\zeta_F(2s) + Ny(z)^{1-s}D^{-\frac{1}{2}}\left(\frac{\pi^{\frac{1}{2}}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\right)^{r_1}\cdot \\ &\quad \left(\frac{2\pi}{2s-1}\right)^{r_2}\zeta_F(2s-1) + 2^{r_1+r_2}D^{-s}\left(\frac{\pi^s}{\Gamma(s)}\right)^{r_1}\left(\frac{(2\pi)^{2s}}{\Gamma(2s)}\right)^{r_2}\cdot \\ &\quad \sum_{\{\mu,\nu\}'}\left|\frac{N\nu}{N\mu}\right|^{s-\frac{1}{2}}e^{2\pi i S(\mu\nu\varpi x)} \times \prod_{j=1}^{r_1+r_2} K_{e_j(s-\frac{1}{2})}(2e_j\pi|(\mu\nu\varpi)_j|y_j)y_j^{\frac{e_j}{2}}. \end{aligned} \tag{1.2.8}$$

where $S(\mu\nu\varpi x) = \sum_{j=1}^{r_1+r_2} e_j \operatorname{Re}((\mu\nu\varpi)_j x_j)$ and the sum is taken over classes of pairs $(\mu,\nu) \in \mathcal{O}\times\mathcal{O}, \mu\nu \neq 0$ under the equivalence relation $(\mu,\nu) \cong (\mu\epsilon, \nu\epsilon^{-1})$ for a unit ϵ.

The modified Bessel function K of second kind is given by

$$2\left|\frac{b}{a}\right|^u K_u(2|ab|) = \int_0^\infty e^{-(a^2t+b^2t^{-1})}t^u\frac{dt}{t}, \tag{1.2.9}$$

for any nonzero real numbers a, b.

As a result, we can write down a limit formula at $s = 1$ and obtain the following function:

$$\begin{aligned} h(z) = \frac{wD}{2^{n-1}\pi^n R}\zeta_F(2)Ny(z) + \frac{2^{r_2+1}w}{R}\sum_{\{\mu,\nu\}'}\left|\frac{N\nu}{N\mu}\right|^{\frac{1}{2}}e^{2\pi i S(\mu\nu\varpi x)}\cdot \\ \prod_{j=1}^{r_1+r_2} K_{\frac{e_j}{2}}(2e_j\pi|(\mu\nu\varpi)_j|y_j)y_j^{\frac{e_j}{2}}, \end{aligned} \tag{1.2.10}$$

where R is the regulator and w is the number of roots of unity in F.

Remark 1.2.1. Let us recover $\log|\eta(z)|$ from the Asai function $h(z)$.

Assume $F = \mathbb{Q}$. We know $D = 1, w = 2, n = 1$ and $R = 1$. Now $S(\mu\nu x) = \mu\nu x$, as $\mu, \nu \in \mathbb{Z}\backslash\{0\}$ and $x \in \mathbb{R}$. Recall that the Bessel function $K_u(x)$ has the following simple expression (e.g. [EMOT53, p.10 (42)]):

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}, \quad x > 0. \tag{1.2.11}$$

Substituting the above data to (1.2.10), we get the following expression:

$$h(z) = \frac{\pi}{3}y + 2\sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty}\mu^{-1}\big(e^{2\pi i\mu\nu z} + e^{-2\pi i\mu\nu\bar{z}}\big), \tag{1.2.12}$$

which is $-4\log|\eta(z)|$.

According to Asai, the function $h(z)$ is analogous to $\log|\eta(z)|$ in the following ways:

Theorem 1.2.2 ([Asa70, Thm.4])**.**

(1) $h(z)$ is real valued and real analytic function on H. It is annihilated by each of the Laplacian-Beltrami operators[1] *on component spaces of H.*

For a real place, the operator is given in coordinates $z = x + iy$ by:

$$-y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}). \tag{1.2.13}$$

For a complex place, it is given by:

$$-y^2(4\frac{\partial^2}{\partial x\partial\bar{x}} + \frac{\partial^2}{\partial y^2}) + y\frac{\partial}{\partial y}. \tag{1.2.14}$$

(2) $h(z)$ satisfies the following modular transformation property: let

$$J(\sigma, z) = \log\prod_{j=1}^{r_1+r_2}(|\gamma_j x_j + \delta_j|^2 + |\gamma_j|^2 y_j^2)^{e_j}, \tag{1.2.15}$$

[1] Asai's definition differs from the usual convention used here by a minus sign.

then $\forall \sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}), z \in H$, *we have*

$$h(z) = J(\sigma, z) + h(\sigma(z)). \tag{1.2.16}$$

(3) $h(z)$ *is associated to the family of Dirichlet series*

$$L(s, \chi)L(s+1, \chi), \tag{1.2.17}$$

for everywhere unramified Grössencharakters χ *via the Mellin transform.*

Now we explain what it is meant precisely by (3).

Denote the multiplicative group of positive real numbers by $\mathbb{R}_+$. To say a Grössencharakter is everywhere unramified means that it is trivial on the maximal compact subgroup of $\mathbb{A}^{\infty,\times}$. Since F is assumed to have class number one, χ is determined by its restriction on the archimedean components, which factors through a homomorphism

$$\mathcal{O}^\times \backslash (\mathbb{R}_+)^{r_1+r_2} \to \mathbb{C}^\times. \tag{1.2.18}$$

Here $\mathcal{O}^\times$ acts on $(\mathbb{R}_+)^{r_1+r_2}$ by:

$$\epsilon(x_j) \mapsto (|\epsilon^{(j)}|x_j), \quad 1 \le j \le r_1 + r_2, \tag{1.2.19}$$

where $\epsilon^{(j)}$ is the image of ϵ under the j-th embedding.

We restrict our consideration to characters χ that are trivial under $\mathbb{R}_{>0}$ (embedded diagonally). By the unit theorem, the group of these characters is isomorphic to $\mathbb{Z}^{r_1+r_2-1}$. These are the 'Charaktere eines Ideals nach den Einheiten' defined by Hecke in [Hec18, §1]. More precisely, if we write $r = r_1 + r_2 - 1$, then $\chi = \chi_m$ for some $m = (m_1, ..., m_r) \in \mathbb{Z}^r$ where for $y \in (\mathbb{R}_+)^{r+1}$:

$$\chi_m(y) = \prod_{j=1}^{r+1} y_j^{2\pi i \sum_{k=1}^{r} m_k e_j^{(k)}}, \tag{1.2.20}$$

and $e_j^{(k)}$ is the element of the matrix:

$$\begin{pmatrix} \frac{e_1}{n} & \dots & \frac{e_{r+1}}{n} \\ e_1^{(1)} & \dots & e_{r+1}^{(1)} \\ \vdots & & \vdots \\ e_1^{(r)} & \dots & e_{r+1}^{(r)} \end{pmatrix} = \begin{pmatrix} 1 & \log|\epsilon_1^{(1)}| & \dots & \log|\epsilon_r^{(1)}| \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & \log|\epsilon_1^{(r+1)}| & \dots & \log|\epsilon_r^{(r+1)}| \end{pmatrix}^{-1}. \tag{1.2.21}$$

Define a map

$$(\mathbb{R}_+)^{r_1+r_2} \to \mathfrak{H}^{r_1} \times \mathbb{H}^{r_2} \tag{1.2.22}$$

by mapping y_j to $\sqrt{-1}y_j$ if $1 \leq j \leq r_1$ and mapping y_j to $\left(\begin{smallmatrix} & -y_j \\ y_j & \end{smallmatrix}\right)$ if $r_1+1 \leq j \leq r+1$. Now define h_0 by subtracting the constant term in the Fourier expansion from h:

$$h_0(z) = h(z) - \frac{wD}{2^{n-1}\pi^n R}\zeta_F(2)Ny(z), \tag{1.2.23}$$

and by $h_0(y)$ we mean the pull back of $h_0(z)$ under the map (1.2.22).

The Mellin transform of the Asai function is then defined by the following integral:

$$I(s,\chi) = \int_{\mathcal{O}^\times\backslash(\mathbb{R}_+)^{r+1}} h_0(y)\bar{\chi}(y)Ny^s d^\times y. \tag{1.2.24}$$

Notations as before, we can state Thm.5 in [Asa70] in the following way:

Theorem 1.2.3. *The Asai function is associated to the L-function*

$$L(s,\chi) = \sum_{0\neq\mu\in\mathcal{O}} \chi(\mu)|N\mu|^{-s}, \operatorname{Re} s > 1 \tag{1.2.25}$$

in the following way:

$$I(s,\chi) = \frac{w^2}{2^{2r_1+r_2-1}R}\chi(\omega)G(s,\chi)G(s+1,\chi)L(s,\chi)L(s+1,\chi), \tag{1.2.26}$$

where $G(s,\chi)$ is the gamma factor given by:

$$G(s,\chi) = (D \cdot \pi^{-2n} \cdot 2^{-2r_2})^{\frac{s}{2}} \prod_{j=1}^{r_1} \Gamma\Big(\frac{s-\sqrt{-1}\alpha_j}{2}\Big) \prod_{j=r_1+1}^{r+1} \Gamma\Big(s - \frac{\sqrt{-1}\alpha_j}{2}\Big) \tag{1.2.27}$$

with $\alpha_j = 2\pi \sum_{k=1}^{r} m_k e_j^{(k)}$.

Remark 1.2.4. The proof is based on a formula of the modified Bessel function:

$$\int_0^\infty K_u(2at) t^s d^\times t = \frac{1}{4} a^{-s} \Gamma\Big(\frac{s-u}{2}\Big)\Gamma\Big(\frac{s+u}{2}\Big) \tag{1.2.28}$$

for any $a > 0$ and $\operatorname{Re} s > |\operatorname{Re} u|$.

1.3 Characterisation of Asai's function and an alternative KLF

We know that the analytic class number formula at $s = 0$ for a number field F with class number one has the following simpler form (which can be deduced from [Neu99, Cor (5.11)]):

$$\zeta_F(s) = -\frac{R}{w} s^{r_1+r_2-1} + O(s^{r_1+r_2}), \tag{1.3.1}$$

compared with that at $s = 1$. Here again R is the regulator and w is the number of roots of unity in F.

Similarly, we can consider the Taylor expansion of Eisenstein series given by (1.2.1) at $s = 0$. It turns out that in so doing, we will obtain a characterisation of Asai's function given by equation (1.2.10) and an alternative KLF, which describes the first order derivative of $E(z,s)$ at $s = 0$ and the proof does not use the computation of the Fourier expansion of Eisenstein series. Section 1.3.1 contains the essence of the method. Section 1.3.2 deals with a more general case which is more complicated mainly because of the presence of complex places.

1.3.1 Case $F = \mathbb{Q}$

Theorem 1.3.1. *Let $F = \mathbb{Q}$ and consider the Eisenstein series as in* (1.2.1)*:*

$$E(z,s) = \sum_{0 \neq (m,n) \in \mathbb{Z}^2} \frac{y^s}{|mz+n|^{2s}}, \operatorname{Re} s > 1. \tag{1.3.2}$$

The following KLF is valid at $s = 0$:

$$E'(z,0) := \frac{d}{ds}\Big|_{s=0} E(z,s) = -\log(y|\eta(z)|^4) - 2\log(2\pi). \tag{1.3.3}$$

Proof. One way to prove this is to manipulate the Fourier expansion of $E(z,s)$, as given in [Sie65]. However, we give an alternative proof, making use of only the constant term of the Fourier expansion of E.

Write the right hand side of (1.3.3) as $\tilde{h}(z)$.

Lemma 1.3.2. *We have the following Taylor expansion of the constant term of $E(z,s)$ at $s = 0$:*

$$c_0(y,s) = -1 + [4\zeta'(0) + \frac{\pi y}{3} - \log y]s + O(s^2). \tag{1.3.4}$$

Proof of Lemma. From the analytic continuation of $E(z,s)$ (for example, in [WMLI92, chap 4]), we know that it is holomorphic at $s = 0$, and that the constant term $c_0(y,s)$ of $E(z,s)$ is:

$$2\zeta(2s)y^s + 2\pi^{\frac{1}{2}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \zeta(2s-1) y^{1-s}. \tag{1.3.5}$$

Now we write down the Taylor expansion of $c_0(y,s)$.

We know

$$\begin{aligned} 2\pi^{\frac{1}{2}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \zeta(2s-1) y^{1-s} &= 2\pi^{\frac{1}{2}} \Gamma(-\frac{1}{2}) \zeta(-1) y s + O(s^2) \\ &= \frac{\pi y}{3} s + O(s^2). \end{aligned} \tag{1.3.6}$$

Also observe:

$$2y^s\zeta(2s) = -1 + (4\zeta'(0) - \log y)s + O(s^2). \tag{1.3.7}$$

Putting together (1.3.6) and (1.3.7), we get the result for $c_0(y, s)$.$\square$

Observe that $\tilde{h}(z)$ satisfies the following properties:

A) $\tilde{h}(z) \to 0$ as $y \to \infty$. This follows from the q-expansion of $\eta(z)$.

B) $\Delta\tilde{h} = -1$ where $\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$.

C) $\tilde{h}(z)$ is invariant under $SL_2(\mathbb{Z})$. This follows from the fact that η^{24} is a modular form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$.

On the other hand, $E'(z, 0)$ also satisfies B) - C). Indeed, it is invariant under $SL_2(\mathbb{Z})$ because $E(z, s)$ is. Since $E(z, s)$ is an eigenfunction of Δ:

$$\Delta E(z, s) = s(1 - s)E(z, s) \tag{1.3.8}$$

applying $\frac{d}{ds}|_{s=0}$ on both sides, we get $\Delta E'(z, 0) = -1$. Finally, from the constant term obtained in Lemma 1.3.2, since $\zeta'(0) = -\frac{1}{2}\log 2\pi$ we see that $E'(z, 0) - \tilde{h}(z) \to 0$ as $y \to \infty$. This means that $E'(z, 0) - \tilde{h}(z)$ is a bounded harmonic function on $\Gamma\backslash\mathfrak{H}$. By the maximum principle, it is a constant hence zero. $\square$

1.3.2 Case F is of class number one

Theorem 1.3.3. *Let $E(z, s)$ denote the Eisenstein series given by* (1.2.8) *and $h(z)$ be the Asai function given by* (1.2.10). *If we write*

$$g(z) := \frac{w}{2^{r_1+r_2}R \cdot (r_1 + r_2)!} \frac{d^{r_1+r_2}}{ds^{r_1+r_2}} E(z, s)\big|_{s=0}, \tag{1.3.9}$$

then the following holds:

$$g(z) = h(z) - \log Ny + C, \tag{1.3.10}$$

where $C = \frac{2w}{R(r_1+r_2)!}\zeta_F^{(r_1+r_2)}(0)$.

To prove the theorem, we need to compute the Taylor expansion of the constant term of $E(z,s)$ first.

Lemma 1.3.4. *The constant term $c_0(y,z)$ of $E(z,s)$ has the following Taylor expansion at $s=0$:*

$$\begin{aligned} c_0(y,s) = &-\frac{2^{r_1+r_2}R}{w}s^{r_1+r_2-1} + \Big\{ -\frac{2^{r_1+r_2}R\log Ny}{w} + \frac{2^{1-r_2}D\zeta_F(2)Ny}{\pi^n} \\ &+ \frac{2^{r_1+r_2+1}}{(r_1+r_2)!}\zeta_F^{(r_1+r_2)}(0)\Big\}s^{r_1+r_2} + O(s^{r_1+r_2+1}). \end{aligned} \tag{1.3.11}$$

Proof. Recall from (1.2.8) that the constant term $c_0(y,s)$ in the Fourier expansion of $E(z,s)$ has the following expression:

$$\begin{aligned} c_0(y,s) = &\,\mathrm{N}\,y(z)^s\zeta_F(2s) + \mathrm{N}\,y(z)^{1-s}D^{-\frac{1}{2}}\left(\frac{\pi^{\frac{1}{2}}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\right)^{r_1} \\ &\left(\frac{2\pi}{2s-1}\right)^{r_2}\zeta_F(2s-1) + 2^{r_1+r_2}D^{-s}\left(\frac{\pi^s}{\Gamma(s)}\right)^{r_1}\left(\frac{(2\pi)^{2s}}{\Gamma(2s)}\right)^{r_2}. \end{aligned} \tag{1.3.12}$$

Now substitute the analytic class number formula at $s=0$ given by (1.3.1), we get the following expression for the first term:

$$\begin{aligned} 2Ny^s\zeta_F(2s) = &-\frac{2^{r_1+r_2}R}{w}s^{r_1+r_2-1} \\ &+ \left\{\frac{2^{r_1+r_2+1}}{(r_1+r_2)!}\zeta_F^{(r_1+r_2)}(0) - \frac{2^{r_1+r_2}R\log Ny}{w}\right\}s^{r_1+r_2} + O(s^{r_1+r_2+1}). \end{aligned} \tag{1.3.13}$$

For the Taylor expansion of the second term, we compute the Taylor expansion of $\zeta_F(2s-1)$ at $s=0$ using the following functional equation for $\zeta_F(s)$ (ref. [Neu99, Cor.(5.10)]):

$$Z_F(s) = Z_F(1-s), \tag{1.3.14}$$

where $Z_F(s) = \zeta_F(s)D^{\frac{s}{2}}\big(\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\big)^{r_1}\big(2(2\pi)^{-s}\Gamma(s)\big)^{r_2}$. The result is:

$$\zeta_F(2s-1) = (-1)^{r_1+r_2}2^{-n}D^{\frac{3}{2}}\pi^{-2r_1-3r_2}\zeta_F(2)s^{r_2} + O(s^{r_2+1}). \tag{1.3.15}$$

Also we can obtain the following expansion at $s = 0$:

$$Ny^{1-s} = Ny - [(\log Ny)Ny]s + O(s^2). \tag{1.3.16}$$

Substituting (1.3.16), (1.3.15) to (1.3.12), we get that the second term is

$$\frac{2^{r_1+r_2-n+1}D\zeta_F(2)Ny}{\pi^n}s^{r_1+r_2} + O(s^{r_1+r_2+1}). \tag{1.3.17}$$

Combining (1.3.13) and (1.3.17), we get the sought-for expression. □

Proof of Theorem. As a result of the Lemma 1.3.4, the constant term of $g(z)$ defined by (1.3.9) is:

$$-\log Ny + \frac{wD\zeta_F(2)}{2^{n-1}\pi^n R}Ny + C. \tag{1.3.18}$$

By the defining equation (1.2.10) of the Asai function $h(z)$, we know that

$$g(z) - h(z) + \log Ny - C \tag{1.3.19}$$

has zero constant term and it tends to 0 exponentially as Ny tends to ∞ as a result of the asymptotic of Bessel functions. By applying $\frac{d^{r_1+r_2}}{ds^{r_1+r_2}}\big|_{s=0}$ to the differential equation satisfied by $E(z,s)$:

$$\Delta_j E(z,s) = e_j^2 s(1-s)E(z,s), 1 \le j \le r_1 + r_2, \tag{1.3.20}$$

we get $\Delta_j g(z) = -e_j^2, 1 \le j \le r_1 + r_2$. This means that (1.3.19) is annihilated by Δ_j for each j. Furthermore, since $h(z) - \log Ny$ is invariant under $\mathrm{SL}_2(\mathcal{O})$ as remarked by Asai[2], the expression (1.3.19) is also invariant under the same group. This implies that for any $\Gamma \subset \mathrm{SL}_2(\mathcal{O})$ such that $\Gamma\backslash H$ is a Riemannian manifold, (1.3.19) is a bounded harmonic function. By the maximum principle, it must be a constant hence zero. □

[2]The proof of the fact that $h(z) - \log Ny$ is invariant under $\mathrm{SL}_2(\mathcal{O})$ without using Eisenstein series is given for $F = \mathbb{Q}(i)$ in [Asa70]. He sketched a similar proof for the general case in the end.

1.A Appendix: Spectral characterisations of Eisenstein series using CdV's method

In [CdV83], Colin de Verdière (CdV) gave an approach of analytic continuation of Eisenstein series using the spectral theory of differential operators. This method is different from that in [Jac72] which uses the Poisson summation formula.

In this section, we will summarise the idea of Colin de Verdière and generalise the method to analytically continue Eisenstein series with weights.

We have decided to include the discussion of CdV's method because we think it is interesting. In fact, the complete method of CdV will imply analytic continuation of Eisenstein series to the whole s-plane. A proper understanding of it would require more functional analysis and we will not mention it in the thesis. We believe that it can be generalised to Eisenstein series with weights defined on totally real fields. However, we will not need it in our discussion of analytic continuation of Eisenstein series in section 4.1 where we follow instead a more well-known method of analysing the intertwining operator and the rest of the constant term.

1.A.1 Summary of CdV's method

To motivate the discussion, let us consider the Eisenstein series given in (1.2.3):

$$E^*(z,s) = \sum_{\Gamma_\infty \backslash \Gamma} y(\gamma z)^s, \ \operatorname{Re} s > 1, \tag{1.A.1}$$

where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_\infty = B(\mathbb{Q}) \cap \Gamma$.

Recall that $\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ is the Laplacian on $\mathfrak{H}$ with respect to the metric $y^{-2}(dx^2 + dy^2)$.

Also recall that $E^*(z,s)$ is an eigenfunction of Δ:

$$\Delta E^*(z,s) = s(1-s)E^*(z,s), \tag{1.A.2}$$

since $\Delta y^s = s(s-1)y^s$ and Δ is Γ-invariant.

By [CdV83], we can treat the analytic continuation in two steps: up to $\operatorname{Re} s > \frac{1}{2}$ and then to the whole complex plane. This implies that one can consider spectral characterisations of E^* in those regions.

Characterisation up to $\operatorname{Re} s = \frac{1}{2}$

Write $X := \Gamma\backslash\mathfrak{H}$. Now let $L^2(X)$ be the space of square integrable functions on X with respect to the inner product:

$$\langle f, g\rangle = \int_X f\bar{g}(z)\frac{dxdy}{y^2},\ \forall f, g \in L^2(X). \tag{1.A.3}$$

Denote by $C^\infty(X)$ the space of smooth functions and $C_c^\infty(X)$ the subspace of functions with compact support. From [CdV83], the following characterisation of $E(z, s)$ is valid:

Theorem 1.A.1. *Fix $b > 1$. For $s \in \mathbb{C}$ with $\operatorname{Re} s > \frac{1}{2}$ and $s \notin [\frac{1}{2}, 1]$, there exists a unique function $F_s(z) \in C^\infty(X)$ such that:*

(1) $\Delta F_s = s(1-s)F_s$,

(2) $F_s - y^s$ *is square integrable on* $\{x + iy \mid 0 \le x \le 1, y \ge b\}$,

(3) $s \mapsto F_s(z)$ *is holomorphic.*

When $\operatorname{Re} s > 1$, $F_s(z) = E(z, s)$.

Remark 1.A.2. This implies that the Eisenstein series $E(z, s)$ can be analytically continued to $\operatorname{Re} s > \frac{1}{2}$ as a holomorphic function in s.

Proof. One can show that the operator Δ is positive, symmetric and densely defined so it has a positive self-adjoint extension by a theorem of Friedrichs in [Fri35].

As an $\mathrm{SL}_2(\mathbb{R})$-invariant differential operator, Δ is only defined on $C_c^\infty(X)$ since some L^2 functions may not be differentiable. It is unbounded, as we can see by taking the eigenfunction $E(z,s)$. The fact that it is symmetric can be seen by applying Green's formula and noticing that X has no boundary.

By definition, the following equality holds:

$$\langle \Delta f, f\rangle = \int_X \left\{ d[(\frac{\partial f}{\partial x}dy - \frac{\partial f}{\partial y}dx)\bar{f}] + (|\frac{\partial f}{\partial x}|^2 + |\frac{\partial f}{\partial y}|^2)\right\} dx \wedge dy. \quad (1.A.4)$$

Since the integration of the first term over X is zero, Δ is positive, i.e.

$$\langle \Delta f, f\rangle \geq 0, \forall f \in C_c^\infty(X). \quad (1.A.5)$$

Now one can apply the extension theorem of Friedrichs in [Fri35]:

Theorem 1.A.3. *A positive, symmetric and densely defined linear operator has a positive self-adjoint extension to the closure of its domain.*

Denote the extension by $\widetilde{\Delta}$. Write $\lambda_s = s(1-s)$. From [RS72, Thm. VIII.2], we know that the resolvent

$$(\widetilde{\Delta} - \lambda_s)^{-1} \quad (1.A.6)$$

is bounded, bijective and holomorphic as an operator-valued function of s, when $\lambda_s \notin [0, +\infty)$.

Now one can construct $F_s(z)$ in the following way. For b given at the beginning of theorem, let $\varphi_{b,b'}(y)$ be the following cut-off function:

$$\varphi_{b,b'}(y) = \begin{cases} 1 & y \geq b \\ 0 & 0 < y \leq b' \end{cases}. \quad (1.A.7)$$

for some b' such that $0 < b' < b$.

Define the following 'pseudo-Eisenstein series' [3].

$$h_s(z) = \sum_{\Gamma_\infty \backslash \Gamma} \varphi_{b,b'}(y(\gamma z))y(\gamma z)^s. \quad (1.A.8)$$

Notice that for any $z \in X$, the sum in $h_s(z)$ is a finite one and so $h_s(z)$ is

[3] This terminology is due to Moeglin-Waldspurger

entire in s. Consider the following function :

$$F_s(z) = h_s(z) - (\widetilde{\Delta} - \lambda_s)^{-1}(\Delta - \lambda_s)h_s(z), \mathrm{Re}\, s > \frac{1}{2}, s \notin [\frac{1}{2}, 1). \qquad (1.A.9)$$

It is not identically zero because $h_s(z) \notin L^2(X)$ when $\mathrm{Re}\, s > \frac{1}{2}$. On the other hand, $(\Delta - \lambda_s)h_s(z) \in L^2(X)$ by construction and the image under $(\widetilde{\Delta} - \lambda_s)^{-1}$ is still so. The uniqueness of $F_s(z)$ follows from bijectivity of $(\widetilde{\Delta} - \lambda_s)^{-1}$. Notice that $F_s(z) = E(z, s)$ when $\mathrm{Re}\, s > 1$.□

1.A.2 Spectral characterisations of Eisenstein series with weights

In this section, we consider $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_\infty = B(\mathbb{Q}) \cap \Gamma$. Cases of other congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ can be studied in a similar way.

Consider the following Eisenstein series defined on $G = \mathrm{SL}_2(\mathbb{R})$ with weight $k \in \mathbb{Z}$:

$$E(g, s)_k = \sum_{\Gamma_\infty \backslash \Gamma} y(\gamma g, s)_k, \quad \mathrm{Re}\, s > 1, \qquad (1.A.10)$$

where

$$y(g, s)_k := y(g(i))^s e^{ik\theta}, \qquad (1.A.11)$$

for $g \in G$ given in Iwasawa coordinates $g = n(x)a(y)h(\theta), n(x) \in N, a(y) \in A, h(\theta) \in SO_2$, where

$$N = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, x \in \mathbb{R} \right\}, A = \left\{ \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix}, y > 0 \right\}. \qquad (1.A.12)$$

We use K to denote SO_2.

In (2.4.65) we will show that this can be recovered from our adelic Eisenstein series.

The group G has a G-invariant Riemannian metric given by

$$ds^2 = y^{-2}(dx^2 + dy^2) + (d\theta - \frac{dx}{2y})^2. \qquad (1.A.13)$$

The associated Laplacian is given by

$$\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) - y\frac{\partial^2}{\partial x \partial \theta} - \frac{5}{4}\frac{\partial^2}{\partial \theta^2}. \tag{1.A.14}$$

Notice that $E(g,s)_k$ is an eigenfunction of Δ with eigenvalue $s(1-s)+\frac{5}{4}k^2$.

We will show that $E(g,s)_k$ also has a spectral characterisation with respect to Δ using the same method from [CdV83].

Denote $\Gamma\backslash G$ by X. Fixing $k \in \mathbb{Z}$, we define a character $\chi_k \colon G \to S^1$ by:

$$\chi_k(g) = e^{ik\theta} \tag{1.A.15}$$

for $g = n(x)a(y)h(\theta)$. For $\gamma \in \Gamma$, we have

$$\chi_k(\gamma g) = e^{ik(\theta + \arg(cz+d))}. \tag{1.A.16}$$

Write $C^\infty(X,\chi_k)$ to denote the space:

$$\{f \colon X \to \mathbb{C} \mid f \text{ is smooth and } f(gh(\theta)) = f(g)e^{ik\theta}\} \tag{1.A.17}$$

It sits in $L^2(X,\chi_k)$, the space of square-integrable functions equipped with the same K-action. Write $C_c^\infty(X,\chi_k)$ and $L_c^2(X,\chi_k)$ to denote the subspaces of functions with compact support in $C^\infty(X,\chi_k)$ and $L^2(X,\chi_k)$ repectively.

Theorem 1.A.4. *Assume $k \neq 0$ and fix $b > 1$. For s in the set :*

$$\left\{ s \in \mathbb{C} \,\middle|\, \operatorname{Re} s > \frac{1}{2}, s \notin (\frac{1}{2}, \frac{1+\sqrt{1+5k^2}}{2}] \right\}, \tag{1.A.18}$$

there exists a unique function $F_{k,s}(g) \in C^\infty(X,\chi_k)$ such that:

(1) $\Delta F_{k,s} = [s(1-s) + \frac{5}{4}k^2]F_{k,s}$,

(2) $F_{k,s} - y^s e^{ik\theta}$ *is square integrable on* $\{g = n(x)a(y)h(\theta) \in G \mid 0 \leq x \leq 1, y \geq b, \theta \in [0,2\pi)\}$,

(3) $s \mapsto F_{k,s}$ is holomorphic.

When $\operatorname{Re} s > 1, s \notin (1, \frac{1+\sqrt{1+5k^2}}{2}]$, *we have* $F_{k,s}(g) = E(g,s)_k$. *Thus* $E(g,s)_k$ *has analytic continuation to the region given by* (1.A.18).

Proof. The idea is as same as before. We show that Δ is a positive symmetric operator and thus by Friedrichs' theorem it has a positive self-adjoint extension. Then we construct $F(g,s)_k$ using the resolvent.

The Laplacian Δ is defined on the space $C_c^\infty(X, \chi_k)$ of compactly supported functions on which K acts via χ_k. It is positive and symmetric.

The positivity of Δ can be seen by completing squares:

$$\Delta = -y^2 \frac{\partial^2}{\partial y^2} - (y\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial \theta})^2 - \frac{\partial^2}{\partial \theta^2}. \tag{1.A.19}$$

The symmetry of Δ can be seen by noticing that all summands

$$-y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}), -y\frac{\partial^2}{\partial x \partial \theta}, -\frac{5}{4}\frac{\partial^2}{\partial \theta^2}$$

in (1.A.14) are symmetric.

Now we show that

Lemma 1.A.5. *The space $C_c^\infty(X, \chi_k)$ is dense in $L^2(X, \chi_k)$.*

Proof of Lemma. Let $f \in L_c^2(X, \chi_k)$. We will construct a family of functions $f_r(g) \in C_c^\infty(X, \chi_k), r \geq 0$ which tends to f in the L^2-norm as $r \to 0$.

For $r \geq 0$, take a smooth function $\varphi_r : \mathbb{R} \to \mathbb{R}$ given by:

$$\varphi_r(x) = \begin{cases} 1 & x \in [1-r, 1+r], \\ 0 & x \in (-\infty, 1-2r] \cup [1+2r, \infty). \end{cases} \tag{1.A.20}$$

Define ϵ_r to be the following function:

$$\epsilon_r(k_1 a(x) k_2) = \varphi_r(x), \quad \forall k_1, k_2 \in K, x > 0. \tag{1.A.21}$$

We know the following decomposition holds for G (ref. [Kir95, p 157, (6)]):

$$G = KAK. \tag{1.A.22}$$

So ϵ_r is a well-defined smooth function on G which is bi-invariant under K and has compact support.

Now we consider the convolution:

$$f_r(g) = \int_G \epsilon_r(h^{-1}) f(gh) dh. \tag{1.A.23}$$

By definition, f_r is smooth. For $h(\theta) \in K$, we have

$$\begin{aligned} f_r(gh(\theta)) &= \int_G \epsilon_r(h^{-1}) f(gh(\theta)h) dh \\ &= \int_G \epsilon_r(h^{-1}) f(g(h(\theta)hh(\theta)^{-1})h(\theta)) dh \\ &= e^{ik\theta} \int_G \epsilon_r(h^{-1}) f(gh) dh \\ &= e^{ik\theta} f_r(g). \end{aligned} \tag{1.A.24}$$

We have shown that $f_r \in C_c^\infty(X, \chi_k)$ for all $r \geq 0$. Notice that by construction $\epsilon_r \to \delta_e$ as $r \to 0$, where δ_e is the Dirac delta function supported at 1. This implies that $f_r \to f$ in the L^2-norm as $r \to 0$, i.e. $C_c^\infty(X, \chi_k)$ is dense in $L_c^2(X, \chi_k)$. We know that each L^2 function in $L_c^2(X, \chi_k)$ can be approximated by L^2 functions that are supported away from ∞, which implies that $C_c^\infty(X, \chi_k)$ is in fact dense in $L^2(X, \chi_k)$.$\square$

Now again by Friedrichs' extension theorem Δ has a positive self-adjoint extension $\widetilde{\Delta}$ to $L^2(\Gamma\backslash G, \chi_k)$. Similarly as before, $(\widetilde{\Delta} - \lambda_{s,k})^{-1}$ is bounded and holomorphic as an operator-valued function of s if $\lambda_{s,k} \notin [0, \infty)$. This condition is equivalent to

$$s \in \mathbb{C} \text{ with } s \notin [\frac{1 - \sqrt{1 + 5k^2}}{2}, \frac{1 + \sqrt{1 + 5k^2}}{2}] \text{ and } \operatorname{Re} s \neq \frac{1}{2}. \tag{1.A.25}$$

Take the same cut-off function φ as in (1.A.7) and consider the following pseudo-Eisenstein series with character χ_k:

$$\Psi_{\chi_k}(g,s) = \sum_{\Gamma_\infty \backslash \Gamma} \varphi(y(\gamma g)) y(\gamma g)^s \chi_k(\gamma g). \tag{1.A.26}$$

It is entire in s as the sum is a finite one.

As a result, when

$$s \in \left\{ s \in \mathbb{C} \middle| \operatorname{Re} s > \frac{1}{2}, s \notin (\frac{1}{2}, \frac{1+\sqrt{1+5k^2}}{2}] \right\}, \tag{1.A.27}$$

the differential equation:

$$(\widetilde{\Delta} - \lambda_{s,k})u = -(\Delta - \lambda_{s,k})\Psi_{\chi_k} \tag{1.A.28}$$

has a non-trivial solution $F_{k,s}(g) - \Psi_{\chi_k}(g,s)$ which is unique. And $F_{k,s}(g) = E(g,s)_k$ when $\operatorname{Re} s > 1, s \notin (1, \frac{1+\sqrt{1+5k^2}}{2}]$.

This implies analytic continuation of $E(g,s)_k$ to the region (1.A.18).$\square$

Chapter 2

Eisenstein series on GL_2

From section 2.1 to 2.2.2, we review concepts and terminologies about automorphic forms on GL_2 and admissible representations of GL_2. In section 2.3 we review tools from harmonic analysis on locally compact groups. They will be used in the constructions of Eisenstein series and all computations that follow. Section 2.4.1 and 2.4.2 explain the constructions and give explicit formulae for local sections used in the constructions. Section 2.4.3 recovers a few classical Eisenstein series from our Eisenstein series. The last section 2.4.4 expands the discussion on the map from the Schwartz space to Eisenstein series.

The background materials on automorphic forms and representations are taken from [God70], [Cas73], [Bum97] and [Cog04]. The constructions appeared here come mainly from [Tat67], [Jac72] and [Gar90].

Notations:
We use F to denote either

(i) a number field, which is a finite extension of $\mathbb{Q}$, with ring of integers $\mathcal{O}$, or

(ii) a local field, which is a finite extension of $\mathbb{Q}_p$ with size of the residue field q_v, ring of integers $\mathcal{O}_v$ and unique maximal ideal $\mathfrak{p}_v$. We will omit the subscript v when there is no danger of confusion.

$\mathbb{A}$: adeles of F,

$\mathbb{A}^\infty$: finite adeles of F, For any ring R, $G(R) = \mathrm{GL}_2(R)$. $N(R)$ is the unipotent subgroup of $G(R)$.

$\mathfrak{g}$: complexification of the Lie algebra of $G(\mathbb{R})$,

$\mathfrak{U}(\mathfrak{g})$: universal enveloping algebra,

$\mathfrak{Z}(\mathfrak{g})$: centre of the universal enveloping algebra,

By a character of a topological group Γ we mean a continuous homomorphism $\Gamma \to \mathbb{C}^\times$ which is not necessarily unitary.

2.1 Automorphic forms

First we recall the notion of an automorphic form on $G(\mathbb{A})$. A *smooth function* on $G(\mathbb{A})$ is a complex valued function which is C^∞ at archimedean places and is locally constant at nonarchimedean places.

Definition 2.1.1. *For $v \mid \infty$, let K_v be a maximal compact subgroup of $G(F_v)$. For $v \nmid \infty$, $K_v = G(\mathcal{O}_v)$. Write $K = \prod_v K_v$. We say a smooth function φ is an* ***automorphic form*** *on $G(\mathbb{A})$ if it satisfies following conditions for $g \in G(\mathbb{A})$:*

(i) modular, that is $\varphi(\gamma g) = \varphi(g), \forall \gamma \in G(F)$,

(ii) right K-finite, that is, the $\mathbb{C}$-vector space $\{\varphi(gk) | k \in K\}$ is finite dimensional,

(iii) $\mathfrak{Z}(\mathfrak{g})$-finite, that is $\{D\varphi | D \in \mathfrak{Z}(\mathfrak{g})\}$ is a finite dimensional $\mathbb{C}$-vector space,

(iv) moderate growth, that is for any norm $|| \quad ||$ on $G(\mathbb{A})$, there exists a positive integer r and a number C such that

$$|\varphi(g)| \leq C||g||^r, \forall g \in G(\mathbb{A}). \tag{2.1.1}$$

We can take for example $||g|| = \prod_v \max_{i,j}\{|g_{i,j}|_v, |(g^{-1})_{i,j}|_v\}$.

Remark 2.1.2. Although this definition is stated for GL_2, it is valid if we replace GL_2 by a reductive group. See [BJ79].

Definition 2.1.3. *The **constant term** of an automorphic form φ is defined to be the following function on $G(\mathbb{A})$:*

$$g \mapsto \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)dn. \tag{2.1.2}$$

*If the constant term of φ is identically zero, then we call it a **cusp form**.*

An important example of automorphic forms on $\mathrm{GL}_1(\mathbb{A})$ is given by the *Hecke character*.

Definition 2.1.4. *A **Hecke character** is a continuous homomorphism $F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$.*

Proposition 2.1.5. *We say a Hecke character is **unramified** at a nonarchimedean place v if its restriction to $\mathcal{O}_v^\times$ is trivial. Otherwise we say it is **ramified**. A Hecke character is unramified outside finitely many places.*

Proposition 2.1.5 is a corollary of a general fact about continuous homomorphisms from totally disconnected and locally compact groups to $\mathrm{GL}_n(\mathbb{C})$:

Proposition 2.1.6. *Any non-trivial continuous homomorphism from a totally disconnected and locally compact group G to $\mathrm{GL}_n(\mathbb{C})$ factors through a discrete quotient. If in addition, G is compact, then we can replace 'discrete' by 'finite'.*

Proof. The proof appears in many places and we would like to summarise the idea here. The pre-image of a neighborhood of 1 in $\mathrm{GL}_n(\mathbb{C})$ which doesn't contain any proper subgroup of $\mathrm{GL}_n(\mathbb{C})$ is an open neighborhood of 1, so contains an open subgroup of G, whose image in $\mathrm{GL}_n(\mathbb{C})$ is necessarily trivial. So the homomorphism has open kernel, so it factors through a discrete quotient. □

As a result of Proposition (2.1.5), we may define the conductor of a Hecke character:

Definition 2.1.7. *The* ***conductor of a Hecke character*** *is a product of ideals $\mathfrak{p}_n^{n_v}$ such that for each place v, $n_v \geq 0$ is the least integer such that χ is trivial on $1 + \mathfrak{p}_v^{n_v}$. Put $1 + \mathfrak{p}_v^0 = \mathcal{O}_v^\times$.*

Now if χ is a Hecke characters $F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$, then it must be of the following form:

$$\xi \cdot | \quad |^s, \tag{2.1.3}$$

where $s \in \mathbb{C}$, and ξ is unitary. Observe that s and ξ are not unique. However, if we require $s \in \mathbb{R}$, then they are unique.

2.2 Admissible representations

We now give a brief summary of the theory of admissible representations for GL_2 related to our construction of Eisenstein series. For details of the terminologies and theory, see [Bum97, Chap 2,3], [Cas73] and [God70].

We now give the definition of an admissible representation of $G(\mathbb{A})$ (or equivalently an admissible $G(\mathbb{A})$-module.)

For each archimedean place v, let K_v be a maximal compact subgroup of $G(F_v)$ and let $\mathfrak{g}_v$ be the complexified Lie algebra of $G(F_v)$. Let $K_\infty = \prod_{v|\infty} K_v$, and $G_\infty = \prod_{v|\infty} G_v$. Let $\mathfrak{g}_\infty$ be the Lie algebra of G_∞ and write $K = \prod_v K_v$.

Definition 2.2.1 ([Fla79, p.182])**.** *An admissible $G(\mathbb{A})$-module V is a vector space which is both a $(\mathfrak{g}_\infty, K_\infty)$-module and a smooth $G(\mathbb{A}^\infty)$-module such that*

(1) the action of $G(\mathbb{A}^\infty)$ commutes with the action $\mathfrak{g}_\infty$ and K_∞, and

(2) for each class σ of continuous irreducible representations of K, the σ-isotypic component (i.e. the sum of all K-submodules of V which are isomorphic to σ) of V has finite dimension.

Now we explain notions of an admissible (g_v, K_v)-module (v archimedean) and an admissible $G(\mathbb{A}^\infty)$-module.

2.2.1 Archimedean place

To define an admissible $(\mathfrak{g}, K)$-module, we start with a vector space equipped with the right action by K and an action of the complexified Lie algebra $\mathfrak{g}$. For $X \in \mathfrak{g}$, define the action of X on $f \in V$ by:

$$Xf(g) = \frac{\partial}{\partial t}\bigg|_{t=0} f(ge^{tX}). \tag{2.2.1}$$

Definition 2.2.2. *We say V is a $(\mathfrak{g}, K)$**-module**, if the following conditions hold:*

(1) The space V is an algebraic direct sum of finite dimensional invariant subspaces under the action of K.

(2) For $X \in \mathrm{Lie}(K)$, X acts on $f \in V$ in the same way as above in (2.2.1).

We say V is admissible if in (1) no finite dimensional invariant subspace occurs with infinite multiplicity. It is irreducible if there is no proper invariant subspace which is nonzero under the actions of $\mathfrak{g}$ and K.

2.2.2 Nonarchimedean place

Definition 2.2.3. *Let Γ be a locally profinite group. We say a representation (π, V) of Γ on a $\mathbb{C}$-vector space V is **admissible**, if the following conditions are satisfied:*

(1) $\forall v \in V$, the stabiliser of v defined by $\{g \in \Gamma | \pi(g)v = v\}$ is open in Γ.

(2) For any open subgroup $K \subset \Gamma$, the space
$V^K = \{v \in V | \pi(k)v = v, \forall k \in K\}$ is finite dimensional.

*If only (1) is satisfied, (π, V) is said to be **smooth**.*

This above definition applies to cases when $\Gamma = G(F)$ (F local) or when $\Gamma = G(\mathbb{A}_F^\infty)$ (F global).

Write $\mathcal{B}(\chi_1, \chi_2)$ to denote the space of locally constant functions f on $G(F)$ satisfying:

$$f\left(\begin{pmatrix} a & b \\ & d \end{pmatrix} g\right) = \chi_1(a)\chi_2(d)\left|\frac{a}{d}\right|^{\frac{1}{2}} f(g), \quad \forall \begin{pmatrix} a & b \\ & d \end{pmatrix} \in B(F). \tag{2.2.2}$$

Notice that $G(F)$ acts on $\mathcal{B}(\chi_1, \chi_2)$ by right translation. This $G(F)$ module is indeed an admissible representation. Any irreducible admissible representation of GL_2 which is not one dimensional falls into one of the three types: *principal series*, *special* and *supercuspidal.*

Definition 2.2.4. *If $\mathcal{B}(\chi_1, \chi_2)$ is irreducible, then it is called a principal series representation of $G(F)$.*

If $\mathcal{B}(\chi_1, \chi_2)$ is reducible, then we know the following fact:

Proposition 2.2.5. *An admissible representation $\mathcal{B}(\chi_1, \chi_2)$ is reducible if and only if $\chi_1\chi_2^{-1} = |\quad|^{\pm}$.*

(1) When $\chi_1\chi_2^{-1} = |\quad|^{-1}$, it has a unique one dimensional subrepresentation and the quotient is irreducible.

(2) When $\chi_1\chi_2^{-1} = |\quad|$, it has a unique infinite dimensional subrepresentation of codimension one, which is irreducible

Proof. See [Bum97, Theorem 4.5.1]. □

In either case, denote the irreducible infinite dimensional subrepresentation (or quotient) by $\sigma(\chi_1, \chi_2)$.

Definition 2.2.6. *We call $\sigma(\chi_1, \chi_2)$ a* ***special representation*** *of $G(F)$.*

The third type of irreducible admissible representation is called the **supercuspidal** representation which is not a parabolic induction from Hecke characters. A description of its Kirillov model (see section 3.3.3) will be enough for our purpose.

To connect local representations with global ones, we need to define the notion of an unramified representation.

Definition 2.2.7. *An admissible irreducible representation of $G(F)$ is called* ***unramified*** *if it contains a $G(\mathcal{O})$-fixed vector.*

Then we need the notion of a restricted tensor product.

Definition 2.2.8 ([Fla79, p.180])**.** *Let $\{W_v | v \in \sum\}$ be a family of vector spaces. Let $\sum_0 \subset \sum$ be a finite subset. For each $v \in \sum \setminus \sum_0$, let x_v be a nonzero vector in W_v. For each finite subset S of $\sum$ containing $\sum_0$, let $W_S = \otimes_{v\in S} W_v$, and if $S \subset S'$, let $f_S \colon W_S \to W_{S'}$ be defined by $\otimes_{v\in S} w_v \mapsto \otimes_{v\in S} w_v \otimes_{v\in S'\setminus S} x_v$. Then the* ***restricted tensor product*** *W of the W_v with respect to the x_v is defined to be the direct limit*

$$\lim_{\rightarrow} W_S. \tag{2.2.3}$$

The space W is spanned by elements of the form $w = \otimes' w_v$, where $w_v = x_v$ for all but finitely many v and $\otimes'$ denotes the restricted tensor product.

Now the connection is provided by the following theorem:

Theorem 2.2.9 ([Fla79, Thm.2],[JL70, Prop.9.1])**.** *If π is an irreducible admissible representation of $G(\mathbb{A})$, then there exist:*

i) irreducible admissible representations π_v of $G(F_v)$ for all finite place v, unramified for almost all v,

ii) irreducible admissible $(\mathfrak{g}_v, K_v)$-modules π_v for all archimedean v unique up to isomorphism.

And that π is the restricted tensor product

$$\pi \cong \otimes' \pi_v \tag{2.2.4}$$

with respect to spherical vectors in the unramified π_v.

2.3 Harmonic analysis on locally compact groups

To carry out analysis on functions on $G(\mathbb{A})$, we recall some theory of harmonic analysis on locally compact groups.

We define a measure dt on $\mathbb{A}$ which is a product of local ones $d_v t$ in the following way:

$v \mid \infty$, $d_v t$ is the usual Lebesgue measure on $\mathbb{R}$ or $\mathbb{C}$;

$v \nmid \infty$, $d_v t$ is the unique Haar measure on F_v such that $\mathcal{O}_v$ has volume one.

Based on that, we define a measure $d^\times t$ on $\mathbb{A}^\times$ with:

$$d_v^\times t = \begin{cases} \frac{d_v t}{|t|_v} & v \mid \infty \\ (1 - q_v^{-1})^{-1} \frac{d_v t}{|t|_v} & v \nmid \infty \end{cases} \tag{2.3.1}$$

where $|\ \ |_v$ is the usual absolute value for $v \mid \mathbb{R}$; it is the square of the complex modulus if $v \mid \mathbb{C}$; it is the canonical absolute value on F_v such that a uniformiser has absolute value q_v^{-1}.

Notice that the unit group $\mathcal{O}_v^\times$ has volume one with respect to $d_v^\times t$ ($v \nmid \infty$).

Define a character $\psi\colon \mathbb{A} \to \mathbb{C}^\times$ by putting $\psi = \prod_v \psi_v$ where:

$$\psi_v(x) = \begin{cases} e^{-2\pi i \operatorname{Tr}_{F_v/\mathbb{Q}_p}(x)} & v \mid \infty \\ e^{2\pi i \Lambda_p[\operatorname{Tr}_{F_v/\mathbb{Q}_p}(x)]} & v \nmid \infty \end{cases} \tag{2.3.2}$$

where $\Lambda_p\colon \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}$ takes the p-principal part in the p-adic series expansion of an element in $\mathbb{Q}_p$.

More generally, we also consider shifts of the character ψ by $F^\times$, for example $\psi(\beta^{-1}x)$ for some $\beta \in F^\times$.

The character ψ defined above enjoys the following property:

Lemma 2.3.1. *$\forall x \in \mathbb{A}, \psi(x + \alpha) = \psi(x), \forall \alpha \in F$.*

Proof. [Gar90, p.275]. □

Recall that Hecke characters were introduced before as automorphic forms on $\mathrm{GL}_1(\mathbb{A})$.

For an integrable function f on F_v, define the Fourier transform $\hat{f}\colon F_v \to \mathbb{C}$ by

$$\hat{f}(x) = \int_{F_v} f(y)\psi_v(xy)dy. \tag{2.3.3}$$

Lemma 2.3.2. *For an integrable function φ on $\mathbb{A}$, we have the* ***Poisson summation formula****:*

$$\sum_{\alpha\in F}\varphi(\alpha)=\sum_{\alpha\in F}\hat{\varphi}(\alpha). \tag{2.3.4}$$

provided that all the sums converge.

Proof. [Gar90, p.278]. □

Later we will prove analytic continuation of our Eisenstein series using this summation formula.

Now we introduce the Schwartz-Bruhat space which will become building blocks of our Eisenstein series. This space will be invariant under the Fourier transform and closed under convolution.

Definition 2.3.3. *The* ***Schwartz-Bruhat*** *space $\mathscr{S}(\mathbb{A}^2)$ consists of functions that are linear combinations of functions of the form:*

$$\phi(x)=\prod_v \phi_v(x_v), \tag{2.3.5}$$

where if

$v\mid\infty, \phi_v\in\mathscr{S}(\mathbb{R}^{2m})$*, the space of smooth functions on $\mathbb{R}^{2m}$ with values in $\mathbb{C}$ for which*

$$|f|_{\alpha,\beta}=\sup_{x\in\mathbb{R}^{2m}}|x_1^{\alpha_1}\cdots x_{2m}^{\alpha_{2m}}|\frac{\partial^{\beta_1+\cdots+\beta_{2m}}f}{\partial x_1^{\beta_1}\cdots\partial x_{2m}^{\beta_{2m}}}(x) \tag{2.3.6}$$

is bounded for all $\alpha_i,\beta_i\in\mathbb{N}$. Put $m=1$ or 2 according to whether v is real or complex, or if

$v\nmid\infty, \phi_v\in\mathscr{S}(F_v^2)$*, the space of functions which are locally constant with compact support. For all but finitely many nonarchimedean places v, ϕ_v is the characteristric function supported on $\mathcal{O}_v^2$.*

Example 2.3.4. Now $\mathbb{A}$ is the adeles of $\mathbb{Q}$. Take $\phi=\prod_v\phi_v$ where

$$\phi_v(u,w)=\begin{cases} e^{-\pi(u^2+w^2)} & v\mid\infty \\ \mathrm{char}_{\mathbb{Z}_p\times\mathbb{Z}_p}(u,w) & v\nmid\infty. \end{cases} \tag{2.3.7}$$

This defines an element $\phi \in \mathscr{S}(\mathbb{A}^2)$.

The idea of studying L-functions using the Schwartz space dated back to [Tat67] and it is Jacquet who formulated Eisenstein series using Schwartz functions in the study of L-functions on $\mathrm{GL}_2 \times \mathrm{GL}_2$ in [Jac72]. It is easy to write down an Eisenstein series and its Fourier coefficients (i.e. Whittaker functions) under specific choices of Schwartz functions. This is one of the main advantages of this approach.

2.4 Adelic Eisenstein series

2.4.1 Constructions

Let χ_1, χ_2 be Hecke characters defined in Definition 2.1.4. Jacquet's construction of Eisenstein series begins with considering functions f in the space[1].

$$\mathcal{B}(\chi_1, \chi_2) := \mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}(\chi_1, \chi_2) \otimes \delta_B^{\frac{1}{2}} \tag{2.4.1}$$

which consists of smooth functions $f \colon G(\mathbb{A}) \to \mathbb{C}$ such that

$$f\left(\begin{pmatrix} x_1 & y \\ & x_1 \end{pmatrix} g\right) = \chi_1(x_1)\chi_2(x_2)\Big|\frac{x_1}{x_2}\Big|^{\frac{1}{2}} f(g), \ \ \forall \begin{pmatrix} x_1 & y \\ & x_1 \end{pmatrix} \in B(\mathbb{A}), g \in G(\mathbb{A}). \tag{2.4.2}$$

Here δ_B is the modulus character on $B(\mathbb{A})$ which is the following group homomorphism $B(\mathbb{A}) \to \mathbb{C}^\times$:

$$\delta_B \begin{pmatrix} x_1 & y \\ & x_1 \end{pmatrix} = \Big|\frac{x_1}{x_2}\Big|. \tag{2.4.3}$$

Jacquet's idea is to consider a subspace of $\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}(\chi_1, \chi_2) \otimes \delta_B^{\frac{1}{2}}$ that is constructed using the Schwartz space $\mathscr{S}(\mathbb{A}^2)$.

First, take a decomposable function ϕ in $\mathscr{S}(\mathbb{A}^2)$, i.e. $\phi = \prod_v \phi_v$. For a character χ with complex modulus $|\chi| = |\ |_{\mathbb{A}}^u, u > 1$, define a distribution on

[1] we use Ind to denote unnormalised induction; some authors use $\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}(\chi_1, \chi_2)$ to denote the representation 2.4.2

$\mathscr{S}(\mathbb{A}^2)$:

$$\mathcal{Z}(\chi, \phi) = \int_{\mathbb{A}^\times} \chi(t)\phi(0,t)d^\times t. \tag{2.4.4}$$

Observe that it is convergent under the above conditions. Then we can consider a family of functions f on $G(\mathbb{A})$ associated to pairs of Hecke characters χ_1, χ_2 by setting:

$$f(g) = \mathcal{Z}(|\ |_{\mathbb{A}} \cdot \chi_1\chi_2^{-1}, g \circ \phi)\chi_1(\det g)|\det g|^{\frac{1}{2}}, \tag{2.4.5}$$

assuming $|\chi_1\chi_2^{-1}| = |\ |^u$ with $\operatorname{Re} u > 1$.

From the defining equation of $f(g)$ in (2.4.5) we see $f \in \mathcal{B}(\chi_1, \chi_2)$:

$$\begin{aligned} f\left(\begin{pmatrix} a & b \\ & d \end{pmatrix} g\right) &= \chi_1(ad)|ad|^{\frac{1}{2}}\chi_1(\det g)|\det g|^{\frac{1}{2}} \int_{\mathbb{A}^\times} |t|\chi_1\chi_2^{-1}(t)\phi((0,dt)g)d^\times t \\ &= \chi_1(ad)\chi_1\chi_2^{-1}(d^{-1})\left|\frac{a}{d}\right|^{\frac{1}{2}} f(g) \\ &= \chi_1(a)\chi_2(d)\left|\frac{a}{d}\right|^{\frac{1}{2}} f(g), \end{aligned} \tag{2.4.6}$$

for all $\begin{pmatrix} a & b \\ & d \end{pmatrix} \in B(\mathbb{A})$. In particular,

$$f(\gamma g) = f(g),\ \forall \gamma \in B(F), \tag{2.4.7}$$

Then we can define Eisenstein series $E(g, f)$ as follows:

$$E(g, f) = \sum_{\gamma \in B(F)\backslash G(F)} f(\gamma g). \tag{2.4.8}$$

which is known to converge for $\operatorname{Re} s > 1$ (ref. [Jac72, prop.19.3]).

From now on, we will assume $\chi_1 = |\ |^{s-\frac{1}{2}}\xi_1, \chi_2 = |\ |^{\frac{1}{2}-s}\xi_2$ **in** (2.1.3) **where** ξ_1, ξ_2 **are unitary and** $s \in \mathbb{C}$. **We can always reduce the general case to the above setting by twisting.** Indeed, for $\chi_i = \xi_i|\ |^{s_i}, i = 1, 2$ where each ξ_i is unitary and $s_i \in \mathbb{C}$, we can set $s = \frac{1}{2}(s_1 - s_2 + 1)$ and work with $\chi_i' = \chi_i \cdot \eta, i = 1, 2$ where $\eta := |\ |^{\frac{1}{2}-s-s_2}$.

Then expression for $f(g)$ in (2.4.5) becomes

$$f(g,s,\xi_1,\xi_2,\phi)=\xi_1(\det g)|\det g|^s\int_{\mathbb{A}^\times}|t|^{2s}\rho(t)\phi((0,t)g)d^\times t,\quad \operatorname{Re}s>1 \tag{2.4.9}$$

where $\rho=\xi_1\xi_2^{-1}$.

We will write $f(g)$ if there is no confusion.

Since ϕ is decomposable, $f(g)$ is a product of $f_v(g)$ over all places v, with

$$f_v(g)=\xi_{1,v}(\det g)|\det g|_v^s\int_{F^\times}|t|_v^{2s}\rho_v(t)\phi_v((0,t)g)d^\times t. \tag{2.4.10}$$

By the definition of restricted tensor product of $\mathscr{S}(\mathbb{A}^2)$, f_v is spherical at almost all places v.

As a corollary of the above constructions, we make the following observations:

Corollary 2.4.1. *The map*

$$f\mapsto E(g,f),\operatorname{Re}s>1 \tag{2.4.11}$$

defines a $G(\mathbb{A})$-equivariant map $\mathcal{B}(\chi_1,\chi_2)\to\mathcal{A}(G(F)\backslash G(\mathbb{A}),\chi_1\chi_2)$ where latter is the space of $\chi_1\chi_2$-central automorphic forms.

In fact, we can view E as a map from the Schwartz space $\mathscr{S}(\mathbb{A}^2)$ to the space of $\chi_1\chi_2$-central automorphic forms $\mathcal{A}(G(F)\backslash G(\mathbb{A}),\chi_1\chi_2)$ by composing this map $f\mapsto E(g,f)$ with that in (2.4.5). Section 2.4.4 will give a more detailed discussion of the latter map.

Notice that from

$$B(F)\backslash G(F)\cong\mathbb{P}^1(F), \tag{2.4.12}$$

we have

$$\sum_{\gamma\in B(F)\backslash G(F)}\phi(te_1\gamma g)=\sum\phi(t\eta g),\quad e_1=(0\quad 1), \tag{2.4.13}$$

where the second sum is over coprime pairs of $\eta\in F^2$. Then substituting

(2.4.13) to the definition of Eisenstein series, we get

$$E(g,s,\xi_1,\xi_2,\phi) = \chi_1(\det g)|\det g|^{\frac{1}{2}} \int_{\mathbb{A}^\times/F^\times} |t|\chi_1\chi_2^{-1}(t)\Theta_0(tg,\phi)d^\times t. \tag{2.4.14}$$

where

$$\Theta(g,\phi) = \sum_{\eta \in F^2} \phi(\eta g), \tag{2.4.15}$$

and put $\Theta_0(g,\phi) = \Theta(g,\phi) - \phi(0)$. The convergence of this sum can be deduced from the definition of Schwartz function in definition 2.3.3.

This expression will be useful in proving analytic continuation and functional equation of Eisenstein series, which will make the expression (2.4.14) valid for all $s \in \mathbb{C}$ except perhaps at two points. The analytic continuation will be discussed in detail in section 4.1. We now state the result here:

Theorem 2.4.2 ([Jac72, p. 120]). *The Eisenstein series $E(g,s,\xi_1,\xi_2,\phi)$ has analytic continuation to the whole s-plane with at most two simple poles. It satisfies the following functional equation:*

$$E(g,s,\xi_1,\xi_2,\phi) = E((g^{-1})^t, 1-s, \bar{\xi_1}, \bar{\xi_2}, \hat{\phi}) \tag{2.4.16}$$

where $\hat{\phi}$ is the Fourier transform of ϕ.

2.4.2 Examples of sections

In the next few lemmas, we compute local sections f_v with specific choices of Schwartz functions.

Nonarchimedean places We assume the character ρ to be unramfied for the next few propositions.

Proposition 2.4.3. *At a nonarchimedean place v, we choose the Schwartz function ϕ to be*

$$\mathrm{char}_{\mathcal{O}\times\mathcal{O}} \tag{2.4.17}$$

i.e. characteristic function with support $\mathcal{O}\times\mathcal{O}$. For any $g \in G(F)$, $f(g)$ is not identically zero and $f(gh) = \xi_1(\det h)f(g)$ for all $h \in G(\mathcal{O})$. The *following is also valid:*

$$f(1, s, \xi_1, \xi_2, \phi) = (1 - \rho(\varpi) q^{-2s})^{-1}, \quad \operatorname{Re} s > 1, \tag{2.4.18}$$

i.e. the Euler factor of $L(2s, \rho)$ at v.

If ξ_1 is assumed to be unramified in addition, then $f(g, s, \xi_1, \xi_2, \phi)$ is invariant under $G(\mathcal{O})$.

Proof. Notice that ϕ is invariant under $G(\mathcal{O})$ translation. If ρ is unramified, then from the defining integral of f as in (2.4.10) we see that f is not identically zero and it satisfies $f(gh) = \xi_1(\det h) f(g)$ for all $h \in G(\mathcal{O}), g \in G(F)$.

Observe:

$$\begin{aligned} f(1, s, \xi_1, \xi_2, \phi) &= \int_{F^\times} \rho(t)|t|^{2s} \phi(0, t) d^\times t \\ &= \int_{\mathcal{O} \setminus \{0\}} |t|^{2s} \rho(t) d^\times t \\ &= \sum_{n=0}^{\infty} (\rho(\varpi) q^{-2s})^n \int_{\mathcal{O}^\times} d^\times t = (1 - \rho(\varpi) q^{-2s})^{-1}, \quad \operatorname{Re} s > 1. \end{aligned} \tag{2.4.19}$$

If both ρ and ξ_1 are unramified, then $\xi_1(\det h) \equiv 1, \forall h \in G(\mathcal{O})$.□

Now we compute cases when nonmaximal open compact subgroups occur. They do not appear to be given explicitly in the literature.

Proposition 2.4.4. *Let*

$$K_0(\mathfrak{p}^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathcal{O}) \middle| c \equiv 0 \mod \mathfrak{p}^n, n > 0 \right\}. \tag{2.4.20}$$

Choose $\phi = \mathrm{char}_{\mathfrak{p}^n \times \mathcal{O}^\times}$. For $\gamma \in G(\mathcal{O})$, the following is valid:

$$f(\gamma) \neq 0 \Leftrightarrow \gamma \in K_0(\mathfrak{p}^n). \tag{2.4.21}$$

Proof. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0(\mathfrak{p}^n)$, then the γ-translate of the support of

Schwartz function ϕ is:

$$(\mathfrak{p}^n \times \mathcal{O}^\times)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a\mathfrak{p}^n + c\mathcal{O}^\times) \times (b\mathfrak{p}^n + d\mathcal{O}^\times) \tag{2.4.22}$$

which is still $\mathfrak{p}^n \times \mathcal{O}^\times$. This means ϕ is invariant under $K_0(\mathfrak{p}^n)$. Hence for any $\gamma \in K_0(\mathfrak{p}^n)$,

$$\begin{aligned} f(\gamma) &= \xi_1(\det\gamma)\int_F \rho(t)|t|^{2s}\phi(0,t)d^\times t \\ &= \xi_1(\det\gamma)\int_{\mathcal{O}^\times} d^\times t = \xi_1(\det\gamma). \end{aligned} \tag{2.4.23}$$

On the other hand, if $\gamma \in G(\mathcal{O})$, then

$$f(\gamma) = \xi_1(\det\gamma)\int_{F^\times} |t|^{2s}\rho(t)\phi(ct,dt)d^\times t. \tag{2.4.24}$$

Since $(c,d) = 1$, either c or d is a unit. Let us assume $d \in \mathcal{O}^\times$. Then the above expression becomes:

$$\xi_1(\det\gamma)\int_{F^\times} |t|^{2s}\rho(t)\phi(\frac{c}{d}t,t)d^\times t. \tag{2.4.25}$$

By our choice of ϕ, this integral is not zero if and only if the valuation $v(\frac{c}{d}) \geq n$. So $v(c) \geq n$, i.e. $c \in \mathfrak{p}^n$. This means $\gamma \in K_0(\mathfrak{p}^n)$.

If we start with the assumption that $c \in \mathcal{O}^\times$, then by a similar argument, the integral is nonzero if and only if $v(d) \leq -n$. This is impossible since $d \in \mathcal{O}$. $\square$

Proposition 2.4.5. *Let*

$$K_1(\mathfrak{p}^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathcal{O}) \middle| c \equiv 0, d \equiv 1 \mod \mathfrak{p}^n, a \in \mathcal{O}^\times, n > 0 \right\}. \tag{2.4.26}$$

Choose $\phi = \mathrm{char}_{\mathcal{O}\times(\varpi^{-n}+\mathcal{O})}$, *where* ϖ *is any chosen uniformiser. Then* f *is* *invariant under $K_1(\mathfrak{p}^n)$ and we have $\forall \gamma \in K_1(\mathfrak{p}^n)$,*

$$f(\gamma) = \xi_1(\det \gamma)\rho(\varpi^{-n})q^{(2s-1)n}(1-q^{-1})^{-1}. \tag{2.4.27}$$

Proof. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_1(\mathfrak{p}^n)$, we have

$$(\mathcal{O} \times (\varpi^{-n} + \mathcal{O})) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a\mathcal{O} + c(\varpi^{-n} + \mathcal{O})) \times (b\mathcal{O} + d(\varpi^{-n} + \mathcal{O})). \tag{2.4.28}$$

If $a \in \mathcal{O}^\times, d \in 1+\mathfrak{p}^n$, $c \in \mathfrak{p}^n$ and $b \in \mathcal{O}$, then the RHS is still $\mathcal{O} \times (\varpi^{-n} + \mathcal{O})$. Observe:

$$\begin{aligned} f(\gamma) &= \xi_1(\det \gamma) \int_{\varpi^{-n}+\mathcal{O}} |t|^{2s} \rho(t) d^\times t \\ &= \xi_1(\det \gamma)\rho(\varpi^{-n})q^{(2s-1)n}(1-q^{-1})^{-1}. \end{aligned} \tag{2.4.29}$$

□

Remark 2.4.6. Comparing with Proposition 2.4.4, we now have a section which is invariant under the congruence subgroup $K_1(\mathfrak{p}^n)$ but is not zero outside the subgroup.

Ramified places Suppose ρ is ramified at place v. This case does not appear to be given explicitly in the literature.

To construct a nonzero section f, we need to make a proper choice of Schwartz function so that its support lies inside the conductor of ρ.

Proposition 2.4.7. *Assume that ρ has conductor $\mathfrak{p}^n, n > 0$, i.e. $1 + \mathfrak{p}^n$ is the maximal subgroup in $\mathcal{O}^\times$ such that ρ is trivial. Choose a uniformiser ϖ and define the Schwartz function to be*

$$\phi = \mathrm{char}_{\mathcal{O}\times(\varpi^{-\mu}+\mathcal{O})},\ \mu \geq n, \tag{2.4.30}$$

then the following is valid for $x \in F^\times$ with $v(x) = m$:

$$f\begin{pmatrix} x & \\ & 1 \end{pmatrix} = \chi_1(x)|x|^{\frac{1}{2}} f(1) = \xi_1(x)\rho(\varpi)^{-\mu} q^{(2\mu-m)s-\mu}(1-q^{-1})^{-1}. \tag{2.4.31}$$

Proof. Since $f \in \mathcal{B}(\chi_1, \chi_2)$, it suffices to compute $f(1)$:

$$\begin{aligned} f(1) &= \int_{F^\times} |t|^{2s} \rho(t) \phi(0,t) d^\times t \\ &= q^{2s\mu} \rho(\varpi)^{-\mu} \int_{\varpi^{-\mu}+\mathcal{O}} d^\times t \\ &= \rho(\varpi)^{-\mu} q^{(2s-1)\mu} (1-q^{-1})^{-1}. \end{aligned} \tag{2.4.32}$$

Notice that the second equality makes use of the assumption that $\mu \geq n > 0$. So we get (2.4.31).□

Real places If v is real, define $\rho(t) = (\frac{t}{|t|_\mathbb{R}})^\nu, \nu = 0, 1$ as in [Tat67]. Write

$$L_\mathbb{R}(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right). \tag{2.4.33}$$

Lemma 2.4.8 ([Jac72, Lem 17.3.3], [Tat67, 2.5])**.** *At a real place, let* $g \in \mathrm{SL}_2(\mathbb{R})$ *be given in its Iwasawa coordinates* $g = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} h(\theta)$ *where* $x \in \mathbb{R}, y \in \mathbb{R}_+^\times$ *and* $h(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \theta \in [0, 2\pi)$. *Choose Schwartz function* ϕ *as*

$$\phi_k(u,w) = \begin{cases} (-iu+w)^k e^{-\pi(u^2+w^2)}, & k \geq 0, \\ (iu+w)^{-k} e^{-\pi(u^2+w^2)}, & k < 0. \end{cases} \tag{2.4.34}$$

Choose ρ *such that* $\rho(-1)(-1)^{|k|} = 1$, *or equivalently* $\nu \equiv |k| \mod 2$. *Then,*

$$f(g) = L_\mathbb{R}(2s + |k|) e^{ik\theta} y^s. \tag{2.4.35}$$

Proof. Notice that since

$$\phi_k[(u,w)h(\theta)] = \phi_k(u,w) e^{ik\theta}, \tag{2.4.36}$$

we know

$$f(gh(\theta)) = f(g) e^{ik\theta}, \tag{2.4.37}$$

i.e. f is of K-type k.

Since $f \in \mathcal{B}(\chi_1, \chi_2)$, it suffices to compute $f(1)$. By definition,

$$\begin{aligned} f(1) &= \int_{\mathbb{R}^\times} |t|^{2s} \rho(t) \phi_k(0,t) d^\times t \\ &= \int_{\mathbb{R}^\times} |t|^{2s} \rho(t) t^{|k|} e^{-\pi t^2} d^\times t. \end{aligned} \tag{2.4.38}$$

Notice that the integral is nonvanishing if $\rho(-1)(-1)^{|k|} = 1$, in which case we get

$$\begin{aligned} f(1) &= 2 \int_{\mathbb{R}_{\geq 0}} |t|^{2s+|k|} e^{-\pi t^2} d^\times t \\ &= \pi^{-(s+\frac{|k|}{2})} \Gamma(s + \frac{|k|}{2}) \end{aligned} \tag{2.4.39}$$

using the integral representation the gamma function:

$$\Gamma(s) = \int_0^\infty t^s e^{-t} d^\times t, \quad \operatorname{Re} s > 0. \tag{2.4.40}$$

Now for $g \in SL_2(\mathbb{R})$ we know:

$$f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} h(\theta) \right) = \pi^{-(s+\frac{|k|}{2})} \Gamma(s + \frac{|k|}{2}) e^{ik\theta} y^s, \tag{2.4.41}$$

by noticing that ρ is trivial when restricted to $\mathbb{R}_{>0}$. $\square$

We define the local intertwining operator $\mathcal{M}_v$ at a place v (either archimedean or nonarchimedean) via the following integral:

$$\mathcal{M}_v f_v(g) = \int_{F_v} f_v\left(w_0 \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} g \right) du, \ w_0 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \tag{2.4.42}$$

and it converges for s such that $\operatorname{Re} s > \frac{1}{2}$ (ref. [Bum97, prop.2.6.2] and [Bum97, prop.4.5.6]). The global intertwining operator will be discussed in section 3.1.

Now we compute the intertwining operators at real and complex places. For simplicity of notation, we write $\mathcal{M}$ instead of $\mathcal{M}_v$ for the rest of this section.

Lemma 2.4.9. *Assume g and ϕ_k are as given in Lemma 2.4.8, then the following is valid:*

$$\mathcal{M}f(g) = e^{ik\theta}y^{1-s}L_{\mathbb{R}}(2s+|k|)\int_{\mathbb{R}}(u+i)^{-(s+\frac{k}{2})}(u-i)^{-(s-\frac{k}{2})}du. \tag{2.4.43}$$

If we take $k=0$,

$$\mathcal{M}f(g) = y^{1-s}L_{\mathbb{R}}(2s-1). \tag{2.4.44}$$

Proof. Notice that $\mathcal{M}f$ is also of the same K-type as f, so it suffices to consider the case when $h(\theta)=1$ and compute $\mathcal{M}f(1)$. For the moment, assume $k \geq 0$.

By definition,

$$\begin{aligned}
\mathcal{M}f(1) &= \int_{\mathbb{R}} f\left(w_0\begin{pmatrix} 1 & u \\ & 1\end{pmatrix}\right)du \\
&= \int_{\mathbb{R}^\times\times\mathbb{R}} |t|^{2s}\rho(t)\phi_k(-t,-ut)d^\times t du \\
&= \rho(-1)\int_{\mathbb{R}^\times\times\mathbb{R}} |t|^{2s}\rho(t)\phi_k(t,ut)d^\times t du \\
&= \rho(-1)\int_{\mathbb{R}^\times\times\mathbb{R}} |t|^{2s}t^k\rho(t)(u-i)^k\exp(-\pi t^2(1+u^2))d^\times t du \\
&= \rho(-1)\int_{\mathbb{R}^\times\times\mathbb{R}} (1+u^2)^{-s-\frac{k}{2}}(u-i)^k|t|^{2s}t^k\rho(t)e^{-\pi t^2}d^\times t du.
\end{aligned} \tag{2.4.45}$$

The t-integral can be computed directly and it is nonzero if $\rho(-1)(-1)^k=1$:

$$\int_{\mathbb{R}^\times}|t|^{2s}t^k\rho(t)e^{-\pi t^2}d^\times t = \pi^{-(s+\frac{k}{2})}\Gamma(s+\frac{k}{2}). \tag{2.4.46}$$

Combining (2.4.45) and (2.4.46), and make use of the fact that $\mathcal{M}f \in \mathcal{B}(\chi_2,\chi_1)$ (e.g. [Bum97, p.479]), we get (2.4.43). For $k<0$, the computation turns out to be the same apart from replacing the Schwartz function by $\phi_k(u,w) = (iu+w)^{-k}e^{-\pi(|u|^2+|w|^2)}$.

If we take $k = 0$ and make use of the formula:

$$\int_{-\infty}^{\infty} \frac{du}{(1+u^2)^s} = \pi^{\frac{1}{2}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \tag{2.4.47}$$

we get (2.4.44). □

Complex places If v is complex, we follow [Tat67, 2.5] and define $\rho(t) = |t|^{2\lambda}(\frac{t}{|t|})^k, k \in \mathbb{Z}$, λ purely imaginary and $|\ |$ the usual complex modulus. Write:

$$L_{\mathbb{C}}(s) := (2\pi)^{1-s}\Gamma(s). \tag{2.4.48}$$

The following is valid at a complex place:

Lemma 2.4.10 ([Jac72, Lem 18.4], [Tat67, 2.5])**.** *Choose*

$$\phi(u, w) = e^{-2\pi(|u|^2+|w|^2)}, \tag{2.4.49}$$

then $f(g)$ is invariant under K. For $g \in \mathrm{SL}_2(\mathbb{C})$ of the following form

$$g = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix}, \quad x \in \mathbb{C}, y > 0, \tag{2.4.50}$$

the following is valid:

$$f(g) = L_{\mathbb{C}}(2s+\lambda)y^{2s+\lambda}. \tag{2.4.51}$$

Proof. Using polar coordinates, we have:

$$\begin{aligned} f(1) &= \int_{\mathbb{C}\setminus\{0\}} |t|^{4s+2\lambda-k} t^k e^{-2\pi|t|^2} d^\times t \\ &= \int_0^{2\pi}\int_0^{\infty} r^{4s+2\lambda} e^{ik\theta} e^{-2\pi r^2} d^\times r d\theta \end{aligned} \tag{2.4.52}$$

Observe that it is identically zero unless $k = 0$, in which case it is

$$\pi^{1-(2s+\lambda)}\Gamma(2s+\lambda). \tag{2.4.53}$$

Applying the defining property of the space $\mathcal{B}(\chi_1, \chi_2)$ we get the sought-for expression for $f(g)$. $\square$

Regarding the intertwining operator, we have the following result:

Lemma 2.4.11. *For g as before, the intertwining operator $\mathcal{M}f$ is not identically zero if $k = 0$, in which case we have:*

$$\mathcal{M}f(g) = L_{\mathbb{C}}(2s + \lambda - 1)y^{2-2s-\lambda}. \tag{2.4.54}$$

Proof. Since $\mathcal{M}f \in \mathcal{B}(\chi_2, \chi_1)$, we start with computing $\mathcal{M}f(1)$.

By definition:

$$\begin{aligned} &\int_{\mathbb{C}} f\left(w_0 \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\right) du \\ &= \int_{\mathbb{C}\times\mathbb{C}^\times} |t|^{4s} \rho(t)\phi(-t, -ut) du d^\times t \\ &= (-1)^k \int_{\mathbb{C}\times\mathbb{C}^\times} |t|^{4s+2\lambda-2-k} t^k e^{-2\pi(|t|^2+|u|^2)} du d^\times t. \end{aligned} \tag{2.4.55}$$

Notice that the u-integral:

$$\int_{\mathbb{C}} e^{-2\pi|u|^2} du = 1. \tag{2.4.56}$$

Notice that the t-integral is identically zero if $k \neq 0$. So when $k = 0$, we have

$$\begin{aligned} &\int_{\mathbb{C}^\times} |t|^{4s+2\lambda-2} e^{-2\pi|t|^2} d^\times t \\ &= \int_0^{2\pi} \int_0^\infty r^{4s+2\lambda-2} e^{-2\pi r^2} d^\times r d\theta \\ &= \pi^{1-(2s+\lambda-1)} \Gamma(2s + \lambda - 1) \end{aligned} \tag{2.4.57}$$

where $\operatorname{Re} s > \frac{1}{2}$.

Finally, $\mathcal{M}f(g)$ is obtained by applying the defining property of the space $\mathcal{B}(\chi_2, \chi_1)$. $\square$

2.4.3 Relation with classical Eisenstein series

Semi-adelic form

We now rewrite the adelic Eisenstein series as a function on $H \times G(\mathbb{A}^\infty)$, where $H = \mathfrak{H}^{r_1} \times \mathbb{H}^{r_2}$ as in [Asa70]. This will make the connection with classic Eisenstein series more obvious.

Let $g \in G(\mathbb{A})$ with $g = g_\infty g^\infty$ where $g^\infty \in G(\mathbb{A}^\infty)$ and $g_\infty \in G(\mathbb{R})^{r_1} \times G(\mathbb{C})^{r_2}$.

Now consider the map

$$G(\mathbb{R})^{r_1} \times G(\mathbb{C})^{r_2} \to (\pm\mathfrak{H})^{r_1} \times (\pm\mathbb{H})^{r_2} \tag{2.4.58}$$

defined by $g \mapsto g(i)$. At real places, g acts as the usual Hilbert modular group and at complex places its action is given in section 1.2.

This map is invariant under the action of the centre

$$Z(\mathbb{R}) \times Z(\mathbb{C}). \tag{2.4.59}$$

This implies that the image of (2.4.58) is in fact $(\pm\mathfrak{H})^{r_1} \times \mathbb{H}^{r_2}$.

Now assume $g_v \in N(\mathbb{R})A(\mathbb{R})$ at each real place v, where

$$A(\mathbb{R}) = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \middle| a \in \mathbb{R}_+^\times \right\}. \tag{2.4.60}$$

As a result of the map (2.4.58), $E(g, s, \xi_1, \xi_2, \phi)$ descends to a function on $\mathfrak{H}^{r_1} \times \mathbb{H}^{r_2} \times G(\mathbb{A}^\infty)$ given by:

$$E(z, g^\infty, s, \xi_1, \xi_2, \phi) := E(g_\infty g^\infty, s, \xi_1, \xi_2, \phi),\ z \in \mathfrak{H}^{r_1} \times \mathbb{H}^{r_2}, g^\infty \in G(\mathbb{A}^\infty). \tag{2.4.61}$$

Notice that since

$$B(F)\backslash G(F) \cong (B(F) \cap \mathrm{SL}_2(\mathcal{O}))\backslash \mathrm{SL}_2(\mathcal{O}), \tag{2.4.62}$$

we may rewrite the sum in $E(z, g^\infty, s, \xi_1, \xi_2, \phi)$ to be over the quotient $(B(F) \cap \mathrm{SL}_2(\mathcal{O}))\backslash \mathrm{SL}_2(\mathcal{O})$.

Now we recover the following classical Eisenstein series as an example:

Corollary 2.4.12. *Assume* $F = \mathbb{Q}$*, take* $g = g_\infty g^\infty$ *where* $g^\infty = 1$ *and* $g_\infty = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} h(\theta)$ *where* $x \in \mathbb{R}, y \in \mathbb{R}_+^\times$ *and*

$$h(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \theta \in [0, 2\pi) \tag{2.4.63}$$

as in Lemma (2.4.8). *Write* $z = g_\infty(i)$, $y(g_\infty) = y$ *and*

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \right\}. \tag{2.4.64}$$

Then from (2.4.18) *and* (2.4.35)*, we can recover the following Eisenstein series in the classical context:*

(i)

$$E(g_\infty, 1, s, 1, 1, \phi) = L_\mathbb{R}(2s + |k|)\zeta(2s) \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} y(\gamma g_\infty, s)_k, \tag{2.4.65}$$

where k *is an integer and*

$$y(g_\infty, s)_k = y(g_\infty(i))^s e^{ik\theta} \tag{2.4.66}$$

for g_∞ *given in the Iwasawa coordinates above.*

(ii) If we set $k = 0$*, we get a non-holomorphic Eisenstein series on* $\mathfrak{H}$*:*

$$L_\mathbb{R}(2s)\zeta(2s) \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} y(\gamma z)^s. \tag{2.4.67}$$

(iii) If we assume $k \geq 3$ *and set* $s = \frac{k}{2}$*, we then get* $y^{\frac{k}{2}}$ ***times the holomorphic Eisenstein series of weight*** k*. That is:*

$$L_\mathbb{R}(2k)\zeta(k) y^{\frac{k}{2}} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} j(\gamma, z)^{-k}, \tag{2.4.68}$$

where for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $j(\gamma, z) := cz + d$.

Proof. We will show (i). Write $z' = \gamma g(i)$. From

$$e^{ik\theta} = j(g,i)^k \cdot y^{\frac{k}{2}}. \tag{2.4.69}$$

we know

$$\theta' = \theta + \arg(cz+d), \tag{2.4.70}$$

because

$$\begin{aligned} e^{ik(\theta'-\theta)} &= y(\gamma z)^{\frac{k}{2}} \cdot y^{-\frac{k}{2}} \cdot j(\gamma g, i)^k \cdot j(g,i)^{-k} \\ &= \Big(\frac{cz+d}{|cz+d|}\Big)^k \\ &= e^{ik\arg(cz+d)}, \end{aligned} \tag{2.4.71}$$

assuming θ', θ and $\arg(cz+d)$ all lie in $[0, 2\pi)$. Then from (2.4.18) and (2.4.35), we get (2.4.65). $\square$

2.4.4 Schwartz space and Eisenstein series

Our construction of sections in the admissible representation of $G(\mathbb{A})$ relies on the Schwartz space $\mathscr{S}(\mathbb{A}^2)$. In other words, there is a map:

$$\begin{aligned} \Psi_\chi : \mathscr{S}(\mathbb{A}^2) &\to \mathcal{B}(\chi) \\ \phi &\mapsto f(g, s, \xi_1, \xi_2, \phi) \end{aligned} \tag{2.4.72}$$

where χ is a short hand notation for the pair χ_1, χ_2 and $f(g, s, \xi_1, \xi_2, \phi)$ is defined by (2.4.9). Write $\hat{\chi}$ for the pair (χ_2, χ_1).

It is natural to ask if the map Ψ_χ is surjective, in other words, if every section f can be constructed by a Schwartz function. This is in fact the situation in most of the cases. Let us look at the local situation case by case.

Nonarchimedean place

Lemma 2.4.13. *When $\mathcal{B}(\chi)$ is irreducible, the map Ψ_χ is surjective.*

Proof. Notice that both $\mathscr{S}(F^2)$ and $\mathcal{B}(\chi)$ are $G(F)$ modules and Ψ_χ is equivariant under that action. We only need to make sure that this map is non-zero. This is indeed the case as we can construct a section with ϕ spherical at places where ρ doesn't ramify and choose ϕ to be

$$\mathrm{char}_{\mathcal{O}\times(1+\mathfrak{p}^n\mathcal{O})} \tag{2.4.73}$$

for n large enough at places where ρ ramifies. Indeed, it suffices to take $g = 1$ and examine the defining expression of $f_v(1)$ at a ramified place v:

$$\begin{aligned} f_v(1) &= \int_{F_v} |t|^{2s}\rho(t)\phi(0,t)d^\times t \\ &= \int_{1+\mathfrak{p}^n} \rho(t)d^\times t \\ &= \int_{1+\mathfrak{p}^n} d^\times t, \end{aligned} \tag{2.4.74}$$

where the last equality results from the assumption that n is large enough so that ρ is trivial on $1 + \mathfrak{p}^n$. Now this can be evaluated and the result is $q_v^{-n}(1 - q_v^{-1})^{-1}$. For v unramfied we know from Prop 2.4.3 that the section f_v is nonzero.

This shows that the map Ψ_χ is nonzero which implies it is surjective.□

From Proposition 2.2.5, $\mathcal{B}(\chi)$ is reducible if and only if $\chi_1\chi_2^{-1} = |\quad|^{\pm}$.

Lemma 2.4.14. *Suppose $\chi_1\chi_2^{-1} = |\quad|$. Then the map Ψ_χ is surjective.*

Proof. If we write $\chi_1 = \xi|\ |^{s-\frac{1}{2}}, \chi_2 = \xi|\ |^{\frac{1}{2}-s}$, where ξ is unitary. Then from [JL70, p.97-98] and [GJ79, p.226] we know that the map Ψ_χ is surjective when $\mathrm{Re}\, s > 0$. In particular, this holds when $s = 1$. □

Lemma 2.4.15. *When $\chi_1\chi_2^{-1} = |\quad|^{-1}$, then the image of $\mathscr{S}(F^2)$ under Ψ_χ is the one dimensional space spanned by the function $g \mapsto \chi_1(\det g)|\det g|^{\frac{1}{2}}$.*

Proof. From Lemma 2.4.14, the map $\Psi_{\hat\chi}$ is surjective, i.e.

$$\Psi_{\hat\chi}\colon \mathscr{S}(F^2) \twoheadrightarrow \mathcal{B}(\chi_2, \chi_1). \tag{2.4.75}$$

On the other hand, it is valid in general that for $f(g, s, \xi_1, \xi_2, \phi) \in \mathcal{B}(\chi_1, \chi_2)$, the function $\mathcal{M}f \in \mathcal{B}(\chi_2, \chi_1)$ is a product of the function

$$f(g, 1-s, \bar{\xi_1}, \bar{\xi_2}, \hat{\phi}) \tag{2.4.76}$$

and some factor only depending on $\chi_1\chi_2^{-1}$ and the character ψ (ref. [GJ79, p.226]).

Moreover, when $\chi_1\chi_2^{-1} = |\ |^{-1}$, the image of $\mathcal{M}\colon \mathcal{B}(\chi_2, \chi_1) \to \mathcal{B}(\chi_1, \chi_2)$ is the one dimensional space spanned by the function $g \mapsto \chi_1(\det g)|\det g|^{\frac{1}{2}}$ (ref. [God70, Thm 6] and [GJ79, Rmk (4.13)]). So the image of $\Psi_\chi = \mathcal{M} \circ \Psi_{\hat{\chi}}$ is the above one dimensional space.□

Archimedean place

Similar to the nonarchimedean case in Lemma 2.4.13, one has:

Lemma 2.4.16. *When $B(\chi)$ is irreducible, then the map Ψ_χ is surjective.*

Interestingly, one also has:

Lemma 2.4.17. *When $B(\chi)$ is reducible, then the map Ψ_χ is still surjective.*

In fact, not only is the map Ψ_χ surjective, it is so even when it's restricted to a smaller subspace of $\mathscr{S}(F^2)$. This subspace consists of functions of the form:

$$\phi(u, w) = e^{-\pi(|u|^2+|w|^2)} P(u, w, \bar{u}, \bar{w}), \tag{2.4.77}$$

where P is any polynomial in $u, w, \bar{u}, \bar{w}$. Here u, w take values in $\mathbb{R}$ or $\mathbb{C}$ depending on whether we are at a real place or a complex one. For a proof please refer to [God70, 2.14].

Chapter 3

Fourier expansion of Eisenstein series

The main goal of this chapter is to introduce Fourier expansion of Eisenstein series and to compute in section 3.3 onwards their Fourier coefficients (which are Whittaker functions) in various cases. To prepare for the discussion, we begin with a summary of the theory of Whittaker and Kirillov models. The references to the material in section 3.1 and 3.2 are [God70], [Bum97] and [Sch98].

Notations:
F: Either a number field or local field, as at the beginning of Chapter 2.
ψ will be a character on $\mathbb{A}/F$ whose complex conjugate is given in section 2.3.

3.1 Fourier expansion of Eisenstein series

In this section, F denotes a number field.

As a function of x, $E\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g, s, \xi_1, \xi_2, \phi\right)$ is invariant under translation by F. So there is a Fourier expansion:

$$E\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g, s, \xi_1, \xi_2, \phi\right) = \sum_{\epsilon \in F} c_\epsilon \bar{\psi}(\epsilon x) \tag{3.1.1}$$

where the coefficients $c_\epsilon = c_\epsilon(g, s, \xi_1, \xi_2, \phi)$ is given by the following integral:

$$c_\epsilon = \int_{\mathbb{A}/F} E\left(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} g\right) \psi(\epsilon u) du. \tag{3.1.2}$$

Recall the Bruhat decomposition for $G(F)$:

$$G(F) = B(F) \sqcup B(F) w_0 B(F),\ w_0 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}. \tag{3.1.3}$$

Applying this decomposition to the defining series of E in (2.4.8) and unraveling the integral (3.1.2), we obtain:

Proposition 3.1.1. *The Fourier coefficients are given by the following formulae:*

$$c_0 = f(g) + \int_{\mathbb{A}} f\left(w_0 \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} g\right) du \tag{3.1.4}$$

$$c_\epsilon = \int_{\mathbb{A}} f\left(w_0 \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} g\right) \psi(\epsilon u) du, \quad \epsilon \neq 0 \tag{3.1.5}$$

One calls c_0 the **constant term**.

There is an important map from $\mathcal{B}(\chi_1, \chi_2)$ to $\mathcal{B}(\chi_2, \chi_1)$ which appears in the constant terms of the Fourier expansion of Eisenstein series.

Definition 3.1.2. *The global* ***intertwining operator*** *is a map $\mathcal{M} : \mathcal{B}(\chi_1, \chi_2) \to \mathcal{B}(\chi_2, \chi_1)$ defined by:*

$$f(g) \mapsto \int_{\mathbb{A}} f\left(w_0 \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} g\right) du,\ \mathrm{Re}\, s > \frac{1}{2}, \tag{3.1.6}$$

for some Haar measure du on $\mathbb{A}$ given at the beginning of section 2.3.

From the definition we see that $\mathcal{M}$ is $G(\mathbb{A})$ equivariant. Since f is assumed to be decomposable, we have

$$\mathcal{M}f(g) = \prod_v \mathcal{M}_v f_v(g_v), \tag{3.1.7}$$

with $\mathcal{M}_v f_v \in \mathcal{B}_v(\chi_2, \chi_1)$ for each place v.

We will show in section 4.1.2 that the intertwining operator $\mathcal{M}$ has analytic continuation to the entire s-plane with at most two simple poles.

Now we rewrite the above Fourier coefficients in the form of Whittaker functions.

For $\epsilon \neq 0$, by a change of variable $u \mapsto \epsilon^{-1}u$, we have

$$\begin{aligned} c_\epsilon &= \int_{\mathbb{A}} f\left(w_0 \begin{pmatrix} 1 & \epsilon^{-1}u \\ & 1 \end{pmatrix} g\right)\psi(u)du \\ &= \int_{\mathbb{A}} f\left(\begin{pmatrix} 1 & \\ & \epsilon^{-1} \end{pmatrix} w_0 \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix} g\right)\psi(u)du \\ &= \int_{\mathbb{A}} f\left(w_0 \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix} g\right)\psi(u)du \end{aligned} \tag{3.1.8}$$

Write

$$W(g) = \int_{\mathbb{A}} f\left(w_0 \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} g\right)\psi(u)du, \tag{3.1.9}$$

then the last integral in (3.1.8) is $W_\epsilon(g) = W\left(\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix} g\right)$.

We can rewrite the constant term in terms of the intertwining operator $\mathcal{M}$ defined in definition 3.1.2. Setting $x = 0$, we have :

$$E(g, s, \xi_1, \xi_2, \phi) = f(g) + \mathcal{M}f(g) + \sum_{\epsilon \in F^\times} W\left(\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix} g\right). \tag{3.1.10}$$

3.2 Whittaker and Kirillov models

In the expression of $W_\epsilon(g)$, we may write it as a product of functions on $G(F_v)$ for each place v.

Recall in section 2.2, the representations we consider are different between a nonarchimedean and an archimedean place. This results in some difference in the assumptions for Whittaker and Kirillov models. We will present each case separately, although they are defined in similar ways.

3.2.1 Nonarchimedean place

Briefly speaking, the space of functions $W_v(g)$ is isomorphic to an irreducible admissible representation of $G(F_v)$. This space is called **the Whittaker model associated to** π_v.

Now we give a more precise explanation and definition (ref.[God70, p.I.16-17]).

For now, F is a local field.

Let (π, V) be an irreducible admissible infinite dimensional representation of $G(F)$. If V consists of complex-valued functions on $F^\times$ on which π acts in a way such that for matrix $\left(\begin{smallmatrix} a & b \\ & 1 \end{smallmatrix}\right), a \in F^\times, b \in F$:

$$\pi \begin{pmatrix} a & b \\ & 1 \end{pmatrix} \varphi(x) = \psi(bx)\varphi(ax), \tag{3.2.1}$$

then one calls (π, V) the **Kirillov model associated to** π, denoted by $\mathcal{K}(\pi, \psi)$. The elements are called **Kirillov functions**.

Now let L be the linear functional on $\mathcal{K}(\pi, \psi)$ given by evaluating a vector at the identity:

$$L(\varphi) = \varphi(1). \tag{3.2.2}$$

Define a function on $G(F_v)$ by

$$W_\varphi(g) = L(\pi(g)\varphi), \quad \varphi \in \mathcal{K}(\pi, \psi) \tag{3.2.3}$$

We then see from the defining property of $\mathcal{K}(\pi, \psi)$ that:

$$W\left(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} g\right) = \psi(u)W(g), \quad \forall u \in F. \tag{3.2.4}$$

This function $g \mapsto W(g)$ on $G(F)$ is called a **Whittaker function** and the space of Whittaker functions is called the **Whittaker model attached to** π. The group $G(F)$ acts on the Whittaker model via right translation.

It is a fact that for an infinite dimensional irreducible admissible representation π of $G(F)$, there exists a unique Whittaker model which is isomorphic to π (ref. [God70, p 1.17]).

Furthermore, we see that there is a **one-to-one correspondence between Kirillov and Whittaker model attached to the same representation**. The definition above shows one direction. On the other hand, if we are given a Whittaker model, we can recover the corresponding Kirillov model by restricting Whittaker functions to the group:

$$\left\{ \begin{pmatrix} x & \\ & 1 \end{pmatrix} \middle| x \in F^\times \right\}. \tag{3.2.5}$$

3.2.2 Archimedean place

For this section, $F = \mathbb{R}$ or $\mathbb{C}$. We follow the notations in section 2.2.1. One considers smooth function W on $G(F)$ satisfying:

$$W\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \psi(x)W(g). \tag{3.2.6}$$

$W(g)$ is called **rapidly decreasing** if for g given in Iwasawa coordinates with $\begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} \in A(F)$, we have $y^N W(g) \to 0$ as $y \to \infty$ for any $N > 0$. We say W is **analytic** if it is given by a convergent power series expansion in a neighborhood of every $g \in G(F)$. The following characterisation of Whittaker models at archimedean places suffices for our purposes (ref.[Bum97, Thm 2.8.1] and [JL70, Thm 6.3]):

Theorem 3.2.1. *Let (π, V) be an irreducible admissible $(\mathfrak{g}, K)$-module for $G(F)$. Then there exists at most one space $\mathcal{W}(\pi, \psi)$ of smooth K-finite functions W satisfying* (3.2.6) *with the following properties:*

(i) rapidly decreasing and analytic,

(ii) invariant under actions of $\mathfrak{U}(\mathfrak{g})$ and K,

(iii) the space $\mathcal{W}(\pi, \psi)$ is isomorphic to (π, V) as an admissible $(\mathfrak{g}, K)$-module.

The space $\mathcal{W}(\pi, \psi)$ is called the **Whittaker model** of π.

Remark 3.2.2. From the defining formula (3.1.9), we see that if f is decomposable (recall (2.4.10)), then the corresponding Whittaker function W will satisfy (3.2.4) and (3.2.6) at corresponding places.

3.3 Computing Whittaker functions

We have seen that for Hecke characters $\chi_i = |\ |^{s_i}\xi_i, i = 1, 2$, the admissible representation $\mathcal{B}(\chi_1, \chi_2)$ of $G(\mathbb{A})$ is a restricted tensor product of $\mathcal{B}_v(\chi_1, \chi_2)$ over all places v. If $\mathcal{B}$ is irreducible, then the component $\mathcal{B}_v$ is spherical principal series for almost all places v.

Now we will compute Whittaker functions appearing in the Fourier expansion of the Eisenstein series $E(g, s, \xi_1, \xi_2, \phi)$. We consider the following local cases:

(i) ξ_1, ξ_2 **unramified and f is spherical**. This is explicit in many literature, for example [God70, p.1.52]. We will need the result in our global computation.

(ii) $\xi_1\xi_2^{-1}$ **ramifies**. This does not seem to be explicit in the literature. We will give explicit results in section 3.3.2.

(iii) **real and complex places**. Partial results are given in [Dri86, 5]. We will give more general results here in section 3.3.4.

Now we consider the above cases in order.

3.3.1 Unramified places

Notice that if $f \in \mathcal{B}(\chi_1, \chi_2)$ is spherical, then the Whittaker function W associated to f is also spherical.

Proposition 3.3.1 ([God70, p.1.52]). *Suppose χ_1, χ_2 are unramified at v, and ψ has conductor $-\delta \leq 0$, i.e. $\varpi^{-\delta}\mathcal{O}$ is the largest ideal of F on which ψ is trivial. Then there is a unique spherical Whittaker function which is 1 at the identity and has the following valuation:*

$$W\begin{pmatrix} \varpi^m & \\ & 1 \end{pmatrix} = \begin{cases} q^{-\frac{m}{2}} \frac{\alpha_1^{m+1} - \alpha_2^{m+1}}{\alpha_1 - \alpha_2} & m \geq -\delta \\ 0 & m < -\delta \end{cases} \tag{3.3.1}$$

where α_i are Satake parameters given by $\chi_i(\varpi), i = 1, 2$.

Lemma 3.3.2. *Assume $\delta = 0$, i.e. ψ has conductor $\mathcal{O}$. Under the construction in Proposition 2.4.3, the corresponding Whittaker function W is spherical and $W(1) = 1$.*

Proof of Lemma. By construction, the section f is invariant under $G(\mathcal{O})$ so the corresponding Whittaker function W is also spherical.

From Proposition 2.4.3 we know that

$$f(1) = (1 - q^{-1}\alpha_1\alpha_2^{-1})^{-1}. \tag{3.3.2}$$

To compute $W(1)$, we divide the integral expression into two parts:

$$\begin{aligned} W(1) &= \int_F f\left(w_0 \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\right)\psi(u)du \\ &= \int_{\mathcal{O}} f\left(w_0 \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\right)\psi(u)du + \sum_{l=0}^{\infty} \int_{\mathfrak{p}^{-(l+1)} - \mathfrak{p}^{-l}} f\left(w_0 \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\right)\psi(u)du. \end{aligned} \tag{3.3.3}$$

Since f is right $G(\mathcal{O})$-invariant, the first integral is:

$$f(1) \int_{\mathcal{O}} \psi(u)du = f(1) = (1 - q^{-1}\alpha_1\alpha_2^{-1})^{-1}. \tag{3.3.4}$$

Now we show that all but one of the integrals under the summation sign are zero. First notice that

$$w_0 \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} = \begin{pmatrix} -u^{-1} & 1 \\ & -u \end{pmatrix} \begin{pmatrix} 1 & \\ u^{-1} & 1 \end{pmatrix}. \tag{3.3.5}$$

So

$$f\left(w_0 \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \right) = |u|^{-1} \chi_1^{-1} \chi_2(u) f \begin{pmatrix} 1 & \\ u^{-1} & 1 \end{pmatrix}. \tag{3.3.6}$$

When $v(u) < 0$, $\begin{pmatrix} 1 & \\ u^{-1} & 1 \end{pmatrix} \in G(\mathcal{O})$, so each term in the summation $\sum_{l=0}^{\infty}$ is a function of q times the following integral:

$$\int_{\mathfrak{p}^{-(l+1)} - \mathfrak{p}^{-l}} \psi(u) du. \tag{3.3.7}$$

The integral is zero when $l \geq 1$ as the conductor of ψ is assumed to be $\mathcal{O}$. This leaves us with only one term:

$$\int_{\mathfrak{p}^{-1} - \mathcal{O}} \psi(u) du = - \int_{\mathcal{O}} \psi(u) du = -1. \tag{3.3.8}$$

Now putting things together, we get

$$W(1) = f(1)(1 - q^{-1} \alpha_1 \alpha_2^{-1}) = 1. \tag{3.3.9}$$

□

3.3.2 Ramified places

Background

Recall as in section 2.4.2, **a place v is ramified if ρ is ramified at v**. In this case, we need to carefully choose the Schwartz function ϕ so that ρ is trivial in the integral.

Notice that in the current situation, the representation $\mathcal{B}$ is still irreducible. However the section f will not be spherical anymore. In fact, we can always associate the representation $\mathcal{B}$ with a Kirillov model, no matter if $\mathcal{B}$ is irreducible or not (ref.[God70, p 1.27-28]). Instead of using the construction appeared there, we will proceed directly by choosing Schwartz functions carefully and compute the Whittaker functions which are going to be invariant under some open compact subgroup of $G(\mathcal{O})$. The Whittaker functions thus obtained are nonzero.

Computation

Now let us analyse the **local picture**.

Suppose ρ has conductor $\mathfrak{p}^n, n > 0$, i.e. $1 + \mathfrak{p}^n$ is the maximal subgroup in $\mathcal{O}^\times$ such that ρ is trivial.

For any integer $\mu \geq n+1$, choose a uniformiser ϖ and define the Schwartz function ϕ to be

$$\phi = \mathrm{char}_{(1+\mathfrak{p}^\mu)\times\mathcal{O}} + \mathrm{char}_{\mathcal{O}\times(\varpi^{-\mu}+\mathcal{O})}\,. \tag{3.3.10}$$

The section f constructed using the above Schwartz function is not identically zero and is invariant under the following open compact subgroup of $G(\mathcal{O})$:

$$K_1(\mathfrak{p}^\mu) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}) \middle| \; a \equiv d \equiv 1 \mod \mathfrak{p}^\mu, \;\; c \equiv 0 \mod \mathfrak{p}^\mu \right\}. \tag{3.3.11}$$

This is because ϕ is invariant under right translation by $K_1(\mathfrak{p}^\mu)$.

When $\mu = 1$, we can decompose $G(F)$ in terms of double cosets involving $K_1(\mathfrak{p}^\mu)$:

Lemma 3.3.3 (Iwahori-Bruhat decomposition)**.**

$$G(F) = B(F)K_1(\mathfrak{p}) \sqcup B(F)w_0K_1(\mathfrak{p}). \tag{3.3.12}$$

Proof. Use [Bum97, p.501] and notice that the coset representative of

$K_1(\mathfrak{p})$ in $K_0(\mathfrak{p})$ is of the form

$$\begin{pmatrix} 1 & \\ & d \end{pmatrix}, \quad d \in (\mathbb{Z}/p\mathbb{Z})^\times. \tag{3.3.13}$$

Then make use of the identity:

$$w_0 \begin{pmatrix} 1 & \\ & d \end{pmatrix} = \begin{pmatrix} d & \\ & 1 \end{pmatrix} w_0. \tag{3.3.14}$$

□

When $\mu > 1$, this decomposition no longer holds. In fact, we have more than two double cosets in $G(F)$.

We will compute f, $\mathcal{M}f$ and W at two double cosets:

$$B(F)K_1(\mathfrak{p}^\mu) \quad \text{and} \quad B(F)w_0K_1(\mathfrak{p}^\mu). \tag{3.3.15}$$

One can certainly use the same method for other double cosets but we are content with the above cases in order to write down a KLF when the Eisenstein series is holomorphic.

Throughout Propositions 3.3.4 to 3.3.6, we assume $x \in F$ with valuation $v(x) = m$.

Proposition 3.3.4. *The following valuations are valid:*

$$f\begin{pmatrix} x & \\ & 1 \end{pmatrix} = \chi_1(x)|x|^{\frac{1}{2}}f(1) = \xi_1(x)\rho(\varpi)^{-\mu}q^{(2\mu-m)s-\mu}(1-q^{-1})^{-1}, \tag{3.3.16}$$

and

$$f\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix} w_0\right) = \rho(-1)\xi_1(x)q^{-ms-\mu}(1-q^{-1})^{-1}. \tag{3.3.17}$$

Proof.
Since $f \in \mathcal{B}(\chi_1, \chi_2)$, it suffices to compute two cases:

$$f(1),\ f\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}. \tag{3.3.18}$$

By definition,

$$\begin{aligned} f(1) &= \int_{F^\times} |t|^{2s}\rho(t)\phi(0,t)d^\times t \\ &= q^{2s\mu}\rho(\varpi)^{-\mu}\int_{1+\mathfrak{p}^\mu} d^\times t \\ &= \rho(\varpi)^{-\mu}q^{(2s-1)\mu}(1-q^{-1})^{-1}. \end{aligned} \tag{3.3.19}$$

On the other hand,

$$\begin{aligned} f\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} &= \int_{F^\times} |t|^{2s}\rho(t)\phi(-t,0)d^\times t \\ &= \rho(-1)\int_{F^\times} |t|^{2s}\rho(t)\phi(t,0)d^\times t \\ &= \rho(-1)\int_{1+\mathfrak{p}^\mu} \rho(t)d^\times t \\ &= \rho(-1)\int_{1+\mathfrak{p}^\mu} d^\times t \\ &= \rho(-1)q^{-\mu}(1-q^{-1})^{-1}. \end{aligned} \tag{3.3.20}$$

□

Proposition 3.3.5. *Assuming the analytic continuation of $\mathcal{M}f$ in Proposition 4.1.6, we have the following valuations for the function $\mathcal{M}f$:*

$$\mathcal{M}f\begin{pmatrix} x & \\ & 1 \end{pmatrix} = \xi_2(x)\rho(-1)q^{(s-1)m-\mu}(1-q^{-1})^{-1}. \tag{3.3.21}$$

and

$$\mathcal{M}f\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix} w_0\right) = \xi_2(x)\rho(-\varpi^{-\mu})q^{(2\mu+m)(s-1)}(1-q^{-1})^{-1}. \qquad (3.3.22)$$

Proof. Similar to the argument above, it suffices to compute $\mathcal{M}f(1)$ and $\mathcal{M}f\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$.

By definition:

$$\begin{aligned}
\mathcal{M}f(1) &= \int_{F^\times \times F} |t|^{2s}\rho(t)\phi(-t,-ut)d^\times t du \\
&= \rho(-1)\int_{F^\times \times F} |t|^{2s-1}\rho(t)\phi(t,u)d^\times t du \\
&= \rho(-1)\left\{\int_{(1+\mathfrak{p}^\mu)\times\mathcal{O}} |t|^{2s-1}\rho(t)d^\times t du + \int_{(\mathcal{O}\backslash\{0\})\times(\varpi^{-\mu}+\mathcal{O})} |t|^{2s-1}\rho(t)d^\times t du\right\} \\
&= \rho(-1)q^{-\mu}(1-q^{-1})^{-1}.
\end{aligned} \qquad (3.3.23)$$

Notice that the second integral is zero because the conductor of ρ is properly contained in $\mathcal{O}\backslash\{0\}$.

On the other hand,

$$\begin{aligned}
\mathcal{M}f\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} &= \int_{F^\times \times F} |t|^{2s-1}\rho(t)\phi(u,-t)d^\times t du \\
&= \rho(-1)\int_{F^\times \times F} |t|^{2s-1}\rho(t)\phi(u,t)d^\times t du \\
&= \rho(-1)\left\{\int_{(1+\mathfrak{p}^\mu)\times(\mathcal{O}\backslash\{0\})} |t|^{2s-1}\rho(t)du d^\times t + \int_{\mathcal{O}\times(\varpi^{-\mu}+\mathcal{O})} |t|^{2s-1}\rho(t)du d^\times t\right\} \\
&= \rho(-\varpi^{-\mu})q^{(2s-2)\mu}(1-q^{-1})^{-1}.
\end{aligned} \qquad (3.3.24)$$

Notice that the first integral is zero by the same reasoning.□

Proposition 3.3.6. *We have the following valuations for the Whittaker* *function W:*

$$W\begin{pmatrix} x & \\ & 1 \end{pmatrix} = \rho(-1)\xi_2(x)q^{ms-\mu}(1-q^{-1})^{-1}. \tag{3.3.25}$$

and

$$W\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix} w_0\right) = \rho(-\varpi^{-\mu})\xi_2(x)(1-q^{-1})^{-1}q^{(s-1)(2\mu+m)-\max\{0,-m-\mu-l\}}, \tag{3.3.26}$$

where $\mathfrak{p}^{-l}, l \in \mathbb{Z}$ *is the conductor of* ψ.

Proof. By definition (3.1.9):

$$\begin{aligned} W\begin{pmatrix} x & \\ & 1 \end{pmatrix} &= \int_F f\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right)\psi(u)du \\ &= \xi_1(x)|x|^s \int_{F^\times \times F} |t|^{2s}\rho(t)\phi(-xt,-ut)\psi(u)d^\times tdu \\ &= \rho(-1)\xi_2(x)|x|^{-s} \int_{F^\times \times F} |t|^{2s}\rho(t)\phi(t,ut)\psi(u)d^\times tdu \\ &= \rho(-1)\xi_2(x)|x|^{-s} \int_{F^\times \times F} |t|^{2s-1}\rho(t)\phi(t,u)\psi(ut^{-1})d^\times tdu \\ &= \rho(-1)\xi_2(x)|x|^{-s}\left\{ \int_{(1+\mathfrak{p}^\mu)\times\mathcal{O}} \rho(t)\psi(ut^{-1})d^\times tdu \right. \\ &\left. + \int_{\mathcal{O}\setminus\{0\}\times(\varpi^{-\mu}+\mathcal{O})} |t|^{2s-1}\rho(t)\psi(ut^{-1})d^\times tdu \right\}. \end{aligned} \tag{3.3.27}$$

By our assumption that $\rho(t)$ has conductor $\mathfrak{p}^n$ and $\mu \geq n+1$, we know the second integral is zero and the first one becomes:

$$\int_{(1+\mathfrak{p}^\mu)\times\mathcal{O}} \psi(ut^{-1})d^\times tdu. \tag{3.3.28}$$

The integral is nonzero if and only if $v(ut^{-1}) \geq -l$ and $u \in \mathcal{O}$ in which case the integral is:

$$q^{-\mu}(1-q^{-1})^{-1}. \tag{3.3.29}$$

Subsituting this to (3.3.27), we get (3.3.25).

The situation for the other case is slightly different. By a similar computation, we are led to the following expression:

$$
\begin{aligned}
&W\Big(\begin{pmatrix} x & \\ & 1 \end{pmatrix} w_0\Big) \\
&= \int_F f\Big(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} x & \\ & 1 \end{pmatrix}\Big)\psi(u)du \\
&= \xi_1(x)|x|^s \int_{F^\times \times F} |t|^{2s}\rho(t)\phi(ut, -xt)\psi(u)d^\times tdu \\
&= \xi_1(x)|x|^s \int_{F^\times \times F} |t|^{2s-1}\rho(t)\phi(u, -tx)\psi(ut^{-1})d^\times tdu \\
&= \rho(-x^{-1})\xi_1(x)|x|^{1-s} \int_{F^\times \times F} |t|^{2s-1}\rho(t)\phi(u, t)\psi(-xut^{-1})d^\times tdu \\
&= \rho(-x^{-1})\xi_1(x)|x|^{1-s}\Big\{ \int_{(\mathcal{O}\setminus\{0\})\times(1+\mathfrak{p}^\mu)} \rho(t)|t|^{2s-1}\psi(-xut^{-1})d^\times tdu \\
&+ \int_{(\varpi^{-\mu}+\mathcal{O})\times\mathcal{O}} \rho(t)|t|^{2s-1}\psi(-xut^{-1})d^\times tdu\Big\}.
\end{aligned} \tag{3.3.30}
$$

Again, as $\rho(t)$ has conductor $\mathfrak{p}^n$ and $\mu \geq n+1$, we know the first integral is zero and the second one can be written as the following after changing of variable:

$$q^{(2s-1)\mu}\rho(\varpi^{-\mu}) \int_{(1+\varpi^\mu\mathcal{O})\times\mathcal{O}} \psi(-xu\varpi^\mu t^{-1})d^\times tdu. \tag{3.3.31}$$

Again, the integral is nonzero if and only if $v(-xu\varpi^\mu t^{-1}) \geq -l$ and $u \in \mathcal{O}$ which imply:

$$v(u) \geq \max\{0, -m-\mu-l\}, \tag{3.3.32}$$

in which case we have

$$\int_{(1+\varpi^\mu\mathcal{O})\times\mathcal{O}} \psi(-xu\varpi^\mu t^{-1})d^\times tdu = q^{-\mu}(1-q^{-1})^{-1}q^{-\max\{0,-m-\mu-l\}}. \tag{3.3.33}$$

Substituting this result and (3.3.31) to (3.3.30), we get (3.3.26). □

Remark 3.3.7. If the character ψ_v is given as in (2.3.2), then it has conductor $\mathcal{O}_v$, i.e. $l = 0$.

3.3.3 Supercuspidal case

We know that a supercuspidal representation does not come from parabolic induction from a pair of Hecke characters, i.e. not of the form $\mathcal{B}(\chi_1, \chi_2)$. We mention a description from [God70, p 1.22] that will suffice for our application in computing the Rankin-Selberg integral in Chapter 5.

Theorem 3.3.8 ([God70, p 1.22]). *Let π be an irreducible admissible representation of $G(F)$. It is supercuspidal if the Kirillov model $\mathcal{K}(\pi) = \mathcal{S}(F^\times)$, i.e. all Kirillov functions associated to π are Schwartz functions that vanish around zero.*

In this thesis, we may regard this theorem as the definition of a supercuspidal representation.

3.3.4 Archimedean places

Real place

The formula for $f(g)$ when g is given by Iwasawa coordinates has been given in Lemma 2.4.8.

For Whittaker functions, we have the following formula:

Proposition 3.3.9. *Under the same assumptions in Lemma 2.4.8 the following is valid for $\epsilon \in \mathbb{R}^\times$, $y > 0$ and $x \in \mathbb{R}$:*

$$\begin{aligned} &W\left(\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} h(\theta)\right) \\ &= e^{ik\theta}\rho(-\epsilon^{-1})\xi_1(\epsilon)\bar{\psi}(\epsilon x)|\epsilon y|^{1-s} L_{\mathbb{R}}(2s + |k|) I(s, k, y\epsilon) \end{aligned} \tag{3.3.34}$$

where $I(s, k, y\epsilon)$ denotes the following expression:

$$\int_{\mathbb{R}} (w - i)^{-(s-\frac{k}{2})}(w + i)^{-(s+\frac{k}{2})} e^{-2\pi i y\epsilon w} dw. \tag{3.3.35}$$

Proof. First notice that W has the same K-type as f.

Then notice that

$$\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \epsilon x \\ & 1 \end{pmatrix}\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix}. \tag{3.3.36}$$

So

$$W\left(\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix}\right) = \bar{\psi}(\epsilon x) W\begin{pmatrix} \epsilon y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix}. \tag{3.3.37}$$

It suffices to compute the second term:

$$\begin{aligned} & W\begin{pmatrix} \epsilon y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} \\ & = \int_{\mathbb{R}} f\left[\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\begin{pmatrix} \epsilon y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix}\right]\psi(u)du \\ & = \rho(-1)\xi_1(\epsilon)|\epsilon|^s \int_{\mathbb{R}^\times \times \mathbb{R}} |t|^{2s-1}\rho(t)\phi_k(\epsilon y^{\frac{1}{2}} t, y^{-\frac{1}{2}} u)\psi(ut^{-1})d^\times t du \end{aligned} \tag{3.3.38}$$

Changing variables $t \to ty^{-\frac{1}{2}}\epsilon^{-1}, u \to uy^{\frac{1}{2}}$, the above expression becomes

$$\rho(-\epsilon^{-1})\xi_1(\epsilon)|\epsilon y|^{1-s} \int_{\mathbb{R}^\times \times \mathbb{R}} |t|^{2s-1}\rho(t)\phi_k(t,u)\psi(ut^{-1}y\epsilon)d^\times t du. \tag{3.3.39}$$

Now to proceed, we assume $k \geq 0$. We get:

$$\rho(-\epsilon^{-1})\xi_1(\epsilon)|\epsilon y|^{1-s} \int_{\mathbb{R}^\times \times \mathbb{R}} |t|^{2s-1}\rho(t)(-it+u)^k e^{-2\pi i\epsilon t^{-1} yu - \pi(t^2+u^2)} d^\times t du. \tag{3.3.40}$$

Let $w = ut^{-1}$, the integral becomes:

$$\int_{\mathbb{R}^\times \times \mathbb{R}} |t|^{2s}\rho(t)t^k(w-i)^k e^{-2\pi i\epsilon yw - \pi t^2(1+w^2)} d^\times t dw \tag{3.3.41}$$

The above is identically zero unless $\rho(-1)(-1)^k = 1$. When this is satisfied, we get

$$\begin{aligned} &2\int_0^\infty \int_{w\in\mathbb{R}} t^{2s+k}(w-i)^k e^{-2\pi i\epsilon yw-\pi t^2(1+w^2)}dwd^\times t \\ &=\pi^{-(s+\frac{k}{2})}\Gamma(s+\frac{k}{2})\int_{\mathbb{R}}(w-i)^k(w^2+1)^{-(s+\frac{k}{2})}e^{-2\pi iwy\epsilon}dw. \end{aligned} \tag{3.3.42}$$

Substituting this to (3.3.39) we get

$$\begin{aligned} W\begin{pmatrix} \epsilon y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} =&\rho(-\epsilon^{-1})\xi_1(\epsilon)|\epsilon y|^{1-s}\pi^{-(s+\frac{k}{2})}\Gamma(s+\frac{k}{2}) \\ &\int_{\mathbb{R}}(w-i)^{-(s-\frac{k}{2})}(w+i)^{-(s+\frac{k}{2})}e^{-2\pi iy\epsilon w}dw \end{aligned} \tag{3.3.43}$$

As mentioned earlier, when $k<0$, we replace the Schwartz function by $(iu+w)^{-k}e^{-\pi(u^2+w^2)}$ as in (2.4.34) and get:

$$\begin{aligned} W\begin{pmatrix} \epsilon y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} =&\rho(-\epsilon^{-1})\xi_1(\epsilon)|\epsilon y|^{1-s}\pi^{-(s-\frac{k}{2})}\Gamma(s-\frac{k}{2}) \\ &\int_{\mathbb{R}}(w-i)^{-(s-\frac{k}{2})}(w+i)^{-(s+\frac{k}{2})}e^{-2\pi iy\epsilon w}dw. \end{aligned} \tag{3.3.44}$$

□

The following lemma will simplify the integral $I(s,k,y\epsilon)$ in the expression in Proposition 3.3.9.

Lemma 3.3.10. *For $r\in\mathbb{Z}_{>0}, t\in\mathbb{R}$ and $s\in\mathbb{C}, \operatorname{Re}s>\frac{1}{2}$, write*

$$I(s,r,t)=\int_{\mathbb{R}}(w-i)^{-(s-\frac{r}{2})}(w+i)^{-(s+\frac{r}{2})}e^{-2\pi itw}dw. \tag{3.3.45}$$

We have

$$I(s,r,t)=2(-i)^rL_{\mathbb{R}}^{-1}(2s+r)\sum_{m=0}^{r}\begin{pmatrix} r \\ m \end{pmatrix}(-2\pi)^{-m}\cdot\frac{d^m}{dt^m}\Big[|t|^{s+\frac{r-1}{2}}K_{s+\frac{r-1}{2}}(2\pi|t|)\Big]. \tag{3.3.46}$$

Proof. Notice that

$$(w-i)^{-(s-\frac{r}{2})}(w+i)^{-(s+\frac{r}{2})} = (w^2+1)^{-s-\frac{r}{2}}(w-i)^r. \tag{3.3.47}$$

As $r > 0$,

$$(w-i)^r = \sum_{m=0}^{r} \binom{r}{m} (-i)^{r-m} w^m. \tag{3.3.48}$$

So the integrand in $I(s,r,t)$ becomes:

$$\sum_{m=0}^{r} \binom{r}{m} (-i)^{r-m}(-2\pi i)^{-m} \frac{d^m}{dt^m}\left[(w^2+1)^{-s-\frac{r}{2}} e^{-2\pi i t w}\right]. \tag{3.3.49}$$

Recall the following expression of the Bessel function (ref. [Kub73, p.15]):

$$\int_{\mathbb{R}} \frac{e^{-2\pi i t w}}{(1+w^2)^{\nu}} dw = 2\pi^{\nu}|t|^{\nu-\frac{1}{2}}\Gamma(\nu)^{-1}K_{\nu-\frac{1}{2}}(2\pi|t|),\ 0 \neq t \in \mathbb{R}, \operatorname{Re}\nu > \frac{1}{2}. \tag{3.3.50}$$

Substituting this expression to $I(s,r,t)$, we get the sought-for expression.□

Another way of expressing $I(s,r,t)$ is to use Whittaker function $W_{\lambda,\mu}(z)$, $\lambda, \mu, z \in \mathbb{C}$ which is a solution of the differential equation:

$$\frac{d^2W}{dz^2} + \left(-\frac{1}{4} + \frac{\lambda}{z} + \frac{\frac{1}{4}-\mu^2}{z^2}\right) W = 0. \tag{3.3.51}$$

For a more detailed description of $W_{\lambda,\mu}(z)$, please refer to [Gra07, 9.220.4]. Using the formula [Gra07, 3.385.9], we obtain the following expression for $r \in \mathbb{C}, t \in \mathbb{R}^{\times}, \operatorname{Re} s > \frac{1}{2}$:

$$I(s,r,t) = \begin{cases} \pi^s t^{s-1}\Gamma^{-1}\left(s+\frac{r}{2}\right) W_{\frac{r}{2},\frac{1}{2}-s}(4\pi t), & t > 0, \\ \pi^s (-t)^{s-1}\Gamma^{-1}\left(s-\frac{r}{2}\right) W_{-\frac{r}{2},\frac{1}{2}-s}(-4\pi t), & t < 0. \end{cases} \tag{3.3.52}$$

Complex place

Proposition 3.3.11. *At a complex place v under the same setting as Lemma 2.4.10, W is SU_2-invariant and we have*

$$\begin{aligned} &W\left(\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} k\right) \\ &= (2\pi)\xi_1(\epsilon)\rho(-\epsilon^{-1})(\operatorname{Re}\epsilon)^{2s+\lambda-1}|\epsilon|^{2-2s}\bar{\psi}(\epsilon x)y^{\lambda+1}K_{2s+\lambda-1}(4\pi y\operatorname{Re}\epsilon), \end{aligned} \tag{3.3.53}$$

where $\mathrm{Re}(s) > \frac{1}{8}$.

Proof. It suffices to compute the following:

$$\begin{aligned} &W\begin{pmatrix} \epsilon y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} \\ &= \int_{\mathbb{C}} f\left[\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\begin{pmatrix} \epsilon y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix}\right]\psi(u)du \\ &= \xi_1(\epsilon)|\epsilon|^{2s}\int_{\mathbb{C}^\times\times\mathbb{C}} |t|^{4s}\rho(t)\phi(-\epsilon y^{\frac{1}{2}}t, -y^{-\frac{1}{2}}ut)\psi(u)d^\times tdu \\ &= \xi_1(\epsilon)|\epsilon|^{2s}\rho(-1)\int_{\mathbb{C}^\times\times\mathbb{C}} |t|^{4s-2}\rho(t)\phi(\epsilon y^{\frac{1}{2}}t, y^{-\frac{1}{2}}u)\psi(ut^{-1})d^\times tdu \\ &= \xi_1(\epsilon)\rho(-\epsilon^{-1})|\epsilon y|^{-2s+2}\int_{\mathbb{C}^\times\times\mathbb{C}} |t|^{4s-2}\rho(t)\phi(t,u)\psi(ut^{-1}y\epsilon)d^\times tdu \end{aligned} \tag{3.3.54}$$

Substituting ϕ and ρ, the expression becomes:

$$\xi_1(\epsilon)\rho(-\epsilon^{-1})|\epsilon y|^{-2s+2}\int_{\mathbb{C}^\times\times\mathbb{C}} |t|^{4s+2\lambda-2}e^{-2\pi(|t|^2+|u|^2)-2\pi iy[ut^{-1}\epsilon+\overline{ut^{-1}\epsilon}]}d^\times tdu \tag{3.3.55}$$

Take $w = ut^{-1}$, then the integral becomes:

$$\int_{\mathbb{C}^\times\times\mathbb{C}} |t|^{4s+2\lambda}e^{-2\pi|t|^2(1+|w|^2)-2\pi iy(w\epsilon+\overline{w\epsilon})}d^\times tdw \tag{3.3.56}$$

Change variable $t \mapsto t(1+|w|^2)^{-\frac{1}{2}}$:

$$\int_{\mathbb{C}^\times \times \mathbb{C}} (1+|w|^2)^{-2s-\lambda}|t|^{4s+2\lambda}e^{-2\pi|t|^2-2\pi iy(w\epsilon+\overline{w\epsilon})}d^\times t dw. \tag{3.3.57}$$

Now consider the t-integral:

$$\begin{aligned} &\int_{\mathbb{C}^\times} |t|^{4s+2\lambda}e^{-2\pi|t|^2}d^\times t \\ &= (2\pi)^{-(2s+\lambda-1)}\Gamma(2s+\lambda) \end{aligned} \tag{3.3.58}$$

Now we deal with the w-integral in (3.3.57). Putting $w = re^{i\theta}$ and $\epsilon = \alpha + \beta i$, we have:

$$\begin{aligned} &\int_{\mathbb{C}^\times} (1+|w|^2)^{-2s-\lambda}e^{-2\pi iy(w\epsilon+\overline{w\epsilon})}dw \\ &= \int_0^{2\pi}\int_0^\infty (1+r^2)^{-2s-\lambda}e^{-4\pi iyr(\alpha\cos\theta-\beta\sin\theta)}rdrd\theta. \end{aligned} \tag{3.3.59}$$

Notice that this integral is invariant if we multiply ϵ by a unitary complex number. Therefore, we can assume $\beta = 0$.

Now making use of an integral expression of the Bessel function of the first kind J_0 (ref. [WW96, p.355]):

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta}d\theta, \tag{3.3.60}$$

we get:

$$\begin{aligned} &\int_0^{2\pi}\int_0^\infty (1+r^2)^{-2s-\lambda}e^{-4\pi iyr\alpha\cos\theta}rdrd\theta \\ &= (2\pi)\int_0^\infty r(1+r^2)^{-2s-\lambda}J_0(4\pi y\alpha r)dr. \end{aligned} \tag{3.3.61}$$

Here we use the fact that $J_0(x) = J_0(-x)$ (ref. [WW96, p.357]).

Now make use of a formula which expresses an integral of J_0 in terms of modified Bessel function K_s ([EMOT53, p.95]):

$$\int_0^\infty J_m(bt)(t^2+z^2)^{-s}t^{m+1}dt = (\frac{1}{2}b)^{s-1}\Gamma(s)^{-1}z^{1+m-s}K_{s-m-1}(bz),$$
$$\mathrm{Re}(2s-\frac{1}{2}) > \mathrm{Re}(m) > -1, \mathrm{Re}(z) > 0. \tag{3.3.62}$$

we get when $\mathrm{Re}(4s-\frac{1}{2}) > 0$,

$$\int_0^\infty r(1+r^2)^{-2s-\lambda}J_0(4\pi y\alpha r)dr = (2\pi y\alpha)^{2s+\lambda-1}\Gamma(2s+\lambda)^{-1}K_{2s+\lambda-1}(4\pi y\alpha). \tag{3.3.63}$$

Substituting (3.3.58) and (3.3.62) to (3.3.55), we get (3.3.53).□

Chapter 4

Generalised Asai's functions and KLFs

In this chapter, we start with a summary of analytic continuation of Eisenstein series based on analysing their constant terms. This method can be found in [GS88, II.1.2] and [Bum97, 3.7]. In section 4.2, we generalise the Kronecker limit formula for Eisenstein series in cases when it has a pole at $s = 1$ and when it is holomorphic.

Notations:
F: Either a number field or local field, as the beginning of Chapter 2.
ψ will be a character on $\mathbb{A}/F$ whose complex conjugate is given in section 2.3.

4.1 Analytic continuation of Eisenstein series

From general theory of Eisenstein series (ref. [GS88, II.1.2]), we know that $E(g, s, \xi_1, \xi_2, \phi)$ has analytic continuation to the entire s-plane with possible simple poles coming from the constant term:

$$f(g, s, \xi_1, \xi_2, \phi) + \mathcal{M}f(g, s, \xi_1, \xi_2, \phi), \tag{4.1.1}$$

in the Fourier expansion (3.1.10).

We will explain in section 4.1.1 and 4.1.2 that when $\xi_1\xi_2^{-1} = |\ |^{\lambda}$ with $\operatorname{Re}\lambda = 0$, there are two simple poles from the first term $f(g)$ and two simple poles from the function $\mathcal{M}f(g)$. Upon cancellations (ref. Lemma 4.1.7), only two of them will show up in the end.

4.1.1 Continuation of $f(g)$

Proposition 4.1.1. *When $\rho = |\ |^{\lambda}$ with $\operatorname{Re}\lambda = 0$, $f(g)$ has at most two simple poles which occur at $s = -\frac{\lambda}{2}$ and $s = \frac{1-\lambda}{2}$.*

The analytic continuation of $f(g)$ follows from Tate's method of continuing zeta functions.

Proof. Recall if Φ is a Schwartz function on $\mathbb{A}^{\times}$ and χ a unitary Hecke character, then from [Tat67] the zeta function:

$$\zeta(s,\chi,\Phi) = \int_{\mathbb{A}^{\times}} |t|^s \chi(t)\Phi(t)d^{\times}t, \quad \operatorname{Re}s > 1 \tag{4.1.2}$$

can be analytically continued to the whole s-plane.

Now let us summarise how the proof works.

Write $P := \{t \in \mathbb{A}^{\times} | |t| > 1\}$ and let $\mathbb{A}_1^{\times}$ be the set of ideles with norm 1.

Let

$$\zeta_1(s,\chi,\Phi) = \int_P |t|^s \chi(t)\Phi(t)d^{\times}t. \tag{4.1.3}$$

Then we can express $\zeta(s,\chi,\Phi)$ as follows:

$$\zeta(s,\chi,\Phi) = \begin{cases} \zeta_1(s,\chi,\Phi) + \zeta_1(1-s,\chi^{-1},\hat{\Phi}), & \chi|_{\mathbb{A}_1^{\times}} \neq 1, \\ \zeta_1(s,\chi,\Phi) + \zeta_1(1-s,\chi^{-1},\hat{\Phi}) & \\ -\rho_F\{\frac{\Phi(0)}{s+\lambda} + \frac{\hat{\Phi}(0)}{1-s-\lambda}\}, & \chi(t) = |t|^{\lambda}, \quad \operatorname{Re}(\lambda) = 0. \end{cases} \tag{4.1.4}$$

where $\rho_F = \frac{2^{r_1}(2\pi)^{r_2}hR}{\sqrt{D}w}$, r_1, r_2 being the number of real and complex embeddings of F, h the class number, R the regulator, D the absolute value of the discriminant and w the number of roots of unity.

We now apply this result to a section $f(g,s,\xi_1,\xi_2,\phi) \in \mathcal{B}(\chi_1,\chi_2)$. Notice that the residue of $f(g,s,\xi_1,\xi_2,\phi)$ is dependent on both g and the Schwartz function ϕ.

Write $\phi_g(t) = \phi[(0 \quad t)g]$.

Using the method above we can write:

$$
\begin{aligned}
f(g) = \xi_1(\det g)|\det g|^s \Big\{ & \zeta_1(2s, \rho, \phi_g) + \zeta_1(1-2s, \bar{\rho}, \widehat{\phi}_g) \\
& - \rho_F \Big(\frac{\phi_g(0)}{2s+\lambda} + \frac{\widehat{\phi}_g(0)}{1-2s-\lambda} \Big) \Big\}
\end{aligned}
\tag{4.1.5}
$$

where

$$
\widehat{\phi}_g(t) = \int_{\mathbb{A}} \phi_g(y)\psi(yt)dy, \tag{4.1.6}
$$

where ψ is the character given in (2.3.2). This shows that the proposition indeed holds.□

4.1.2 Continuation of $\mathcal{M}f$

We now explain the following fact:

Proposition 4.1.2. *When $\rho = |\ |^\lambda$ for some purely imaginary number λ, the function $\mathcal{M}f$ has at most two simple poles occurring at $s = \frac{1-\lambda}{2}, 1 - \frac{\lambda}{2}$.*

This results from the fact that for the global intertwining operator $\mathcal{M}$, the poles of $\mathcal{M}f$ if any, must belong to those of $L(2s-1, \rho)$.

To analyse the intertwining operator, we decompose it into a product of local components.

First, as for an unramified places v, the following is valid:

Lemma 4.1.3. *If v is a nonarchimedean place where $\mathcal{B}_v(\chi_1, \chi_2)$ is spherical principal series and $f \in \mathcal{B}_v(\chi_1, \chi_2)$ is a spherical vector, then*

$$
\mathcal{M}f = \frac{1 - q^{-1}\alpha_1\alpha_2^{-1}}{1 - \alpha_1\alpha_2^{-1}} f \tag{4.1.7}
$$

where α_1, α_2 are the Satake parameters associated to $\mathcal{B}_v$.

Proof. [Bum97, Prop 4.6.7].□

As a result, the global intertwining operator can be written as follows:

Lemma 4.1.4.

$$\mathcal{M}f(g) = \frac{L_{S\cup\Sigma_\infty}(2s-1,\rho)}{L_{S\cup\Sigma_\infty}(2s,\rho)} \prod_{v\in S\cup\Sigma_\infty} \mathcal{M}_v f_v \prod_{v\notin S\cup\Sigma_\infty} f_v^0 \tag{4.1.8}$$

where $S = \{v \text{ nonarchimedean } |\rho_v \text{ ramifies}\}$, Σ_∞ is the set of archimedean places, and f_v^0 is a spherical vector in $\mathcal{B}_v(\chi_1,\chi_2)$.

Recall from our construction that if $f_v^0 \in \mathcal{B}_v(\chi_1,\chi_2)$ is spherical, then for any $k \in G(\mathcal{O})$,

$$f_v^0(k) = f_v^0(1) = L_v(2s,\rho). \tag{4.1.9}$$

This implies that in (4.1.8), the term

$$\frac{1}{L_{S\cup\Sigma_\infty}(2s,\rho)} \prod_{v\notin S\cup\Sigma_\infty} f_v^0 \tag{4.1.10}$$

is entire in s.

At a place $v \in S\cup\Sigma_\infty$, we know the following fact:

Lemma 4.1.5. *For each $v \in S\cup\Sigma_\infty$, the expression*

$$\frac{\mathcal{M}f_v(g)}{L_v(2s-1,\rho)} \tag{4.1.11}$$

is entire in s.

Proof. Refer to [Bum97, p 357-358].□

Write $L_S(s,\rho)$ to denote the following partial L-function:

$$\prod_{v\notin S} L_v(s,\rho). \tag{4.1.12}$$

If we write $\mathcal{M}f$ in the following way:

$$\mathcal{M}f(g) = L(2s-1,\rho) \prod_{v\in S\cup\Sigma_\infty} \frac{\mathcal{M}_v f_v}{L_v(2s-1,\rho)} \left(\frac{1}{L_{S\cup\Sigma_\infty}(2s,\rho)} \prod_{v\notin S\cup\Sigma_\infty} f_v^0 \right), \tag{4.1.13}$$

then we can conclude

Proposition 4.1.6. *The function $\mathcal{M}f$ is meromorphic. Its poles, if any, belong to those of $L(2s-1,\rho)$.*

This shows that poles of $\mathcal{M}f$ are as described in Proposition 4.1.2.

4.1.3 Poles of Eisenstein series

Proposition 4.1.7. *The pole at $s=\frac{1-\lambda}{2}$ from $f(g)$ cancels with that from $\mathcal{M}f(g)$. So that the Eisenstein series $E(g,s,\xi_1,\xi_2,\phi)$ can have at most two simple poles. This occurs when $\rho=|\ |^\lambda, \operatorname{Re}\lambda=0$ and the poles are $s=-\frac{\lambda}{2}$ and $s=1-\frac{\lambda}{2}$.*

We give a representation theoretic argument from [Bum97, p 361-362].

Proof. Denote by R the residue of Eisenstein series E at $s=\frac{1-\lambda}{2}$. As $f\in\mathcal{B}(\chi_1,\chi_2)$ and $\mathcal{M}f\in\mathcal{B}(\chi_2,\chi_1)$, by letting s tend to $\frac{1-\lambda}{2}$ from the right (i.e. the region that $\operatorname{Re}(s)>\operatorname{Re}(\frac{1-\lambda}{2})=\frac{1}{2}$), we see that $R\in\mathcal{B}(\chi_1,\chi_1)$.

Now we show $R=0$.

Consider an element in $G(\mathbb{A})$: $w_0\begin{pmatrix} y & \\ & y^{-1}\end{pmatrix}$ where $y\in\mathbb{A}^\times$ and $w_0=\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$.

On one hand, $\forall g\in G(\mathbb{A})$,

$$R\left(w_0\begin{pmatrix} y & \\ & y^{-1}\end{pmatrix}g\right)=R\left(\begin{pmatrix} y & \\ & y^{-1}\end{pmatrix}g\right)=|y|R(g) \tag{4.1.14}$$

since R is $G(F)$-invariant.

On the other hand, since

$$w_0\begin{pmatrix} y & \\ & y^{-1}\end{pmatrix}=\begin{pmatrix} y^{-1} & \\ & y\end{pmatrix}\begin{pmatrix} & 1\\ -1 & \end{pmatrix}, \tag{4.1.15}$$

we have

$$R\left(w_0\begin{pmatrix} y & \\ & y^{-1}\end{pmatrix}g\right)=|y|^{-1}R(g). \tag{4.1.16}$$

Since both equalities hold for any $y\in\mathbb{A}^\times$, we must have $R(g)=0$. This means that there is no pole for $E(g,s,\xi_1,\xi_2,\phi)$ at $s=\frac{1-\lambda}{2}$. The rest of the proposition follows from previous two sections.$\square$

4.2 Kronecker limit formulae

To deduce a KLF, we consider the Laurent expansion of our Eisenstein series at $s = 1-\frac{\lambda}{2}$ (refer to Proposition 4.1.7 for a fact about the poles of Eisenstein series). One might consider deducing a KLF with respect to the other pole but the computation will be similar.

For a pair of Hecke characters χ_1, χ_2, **let S be the set of nonarchimedean places v where $\chi_1\chi_2^{-1}$ ramifies**. In this section, we deduce several limit formulae based on Laurent expansions of the function $\mathcal{M}f$ when the set S of 'bad primes' varies. In fact, at $v \in S$, both the section f_v and the function $\mathcal{M}_v f_v$ are nonspherical. We will consider the following cases separately:

(1) $S = \emptyset$, i.e. every nonarchimedean place is unramified.

(2) S is a finite nonempty set of nonarchimedean places. In this case the Eisenstein series will have no pole.

Notations Let the degree of F be $n = r_1 + 2r_2$ where r_1, r_2 are the number of real and complex places respectively. For $y \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, define the norm $Ny = \prod_{i=1}^{r_1+r_2} y_i^{e_i}$, where $e_i = 1$ or 2 depends on whether it is real or complex place.

For each v real, take $g = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} h(\theta)$, for v complex take $g = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix}$ and for v nonarchimedean and $v \notin S$, take $g = \begin{pmatrix} \varpi^m & \\ & 1 \end{pmatrix}$ for some integer m and some uniformiser ϖ of F_v.

To be able to obtain expressions as explicitly as possible, we make the following choices of Schwartz functions as in section 2.4.2:

$$\phi_v(u,w) = \begin{cases} (-iu+w)^{k_v} e^{-\pi(|u|^2+|w|^2)} & v \text{ real}, \quad k_v \geq 0, \\ (iu+w)^{-k_v} e^{-\pi(|u|^2+|w|^2)} & v \text{ real}, \quad k < 0, \\ e^{-2\pi(|u|^2+|w|^2)} & v \text{ complex}, \\ \mathrm{char}_{\mathcal{O}_v \times \mathcal{O}_v} & v \text{ nonarchimedean and } v \notin S, \\ \mathrm{char}_{(1+\mathfrak{p}^{\mu_v})\times\mathcal{O}_v} + \mathrm{char}_{\mathcal{O}_v \times (\varpi^{-\mu_v}+\mathcal{O}_v)} & v \text{ nonarchimedean and } v \in S. \end{cases} \tag{4.2.1}$$

If $S \neq \emptyset$, then for $v \in S$, we assume $\mu_v \geq n_v + 1$ with $1 + \mathfrak{p}^{n_v}$ being the conductor of ρ_v. From our analysis in section 3.3.2, we see that the evaluation of the Fourier coefficients depends on which component g lies in. We distinguish these two cases by writing

$$S = S_1 \sqcup S_2, \tag{4.2.2}$$

where for primes in S_1, f, $\mathcal{M}f$ and W are evaluated at the component $B(F)K_1(\mathfrak{p}^\mu)$ and for primes in S_2 at the component $B(F)w_0K_1(\mathfrak{p}^\mu)$. **This choice of g applies to Proposition 4.2.11 and section 4.2.2**.

L-factors. Before putting together local computations, we construct local zeta integrals and replace L-factors à la Tate appeared in (4.1.13) at bad primes and archimedean places by those zeta integrals. We choose those zeta integrals in such ways that the quotients of $\mathcal{M}f(g)$ (which are given in section 3.3.2) by those integrals are as simple as possible. It can be seen that the quotients of our zeta integrals with corresponding Tate's L-factors are holomorphic. So no extra poles are created in this process.

Lemma 4.2.1. *At a real place, put $\Phi_k(x) = x^{|k|}e^{-\pi x^2}, k \in \mathbb{Z}$ and $\rho(x) = (\frac{x}{|x|})^\lambda, \lambda = 0, 1$ as in Lemma 2.4.8. Assume that $\rho(-1)(-1)^{|k|} = 1$. Then*

$$\zeta(2s, \rho, \Phi_k) = \int_{\mathbb{R}^\times} |t|^{2s} \rho(t) \Phi_k(t) d^\times t = L_{\mathbb{R}}(2s + |k|). \tag{4.2.3}$$

Proof. Direct computation.□

Remark 4.2.2. Tate's L-factor at a real place is based on the same choice of character ρ but a different choice of Schwartz function:

$$\Phi(x) = \begin{cases} e^{-\pi x^2} & \rho = 1 \\ x e^{-\pi x^2} & \rho = \frac{x}{|x|}. \end{cases} \tag{4.2.4}$$

Lemma 4.2.3. *At a complex place, put* $\Phi(x) = e^{-2\pi|x|^2}$ *and choose* $\rho(x) = |x|^{2\lambda}$*. Then the L-factor is:*

$$\zeta(2s, \rho, \Phi) = L_{\mathbb{C}}(2s + \lambda). \tag{4.2.5}$$

Proof. [Tat67, p.318].□

Lemma 4.2.4. *At a nonarchimedean place, the following is valid:*

(i) For $v \notin S$*, take* $\Phi = \mathrm{char}_{\mathcal{O}}$*, then we have*

$$\zeta(2s, \rho, \Phi) = (1 - \rho(\varpi)q^{2s})^{-1}. \tag{4.2.6}$$

(ii) For $v \in S_1$*, take* $\Phi = \mathrm{char}_{1+\mathfrak{p}^{\mu}}$*, then we have*

$$\zeta(2s - 1, \rho, \Phi) = q^{-\mu}(1 - q^{-1})^{-1}. \tag{4.2.7}$$

(iii) For $v \in S_2$*, take* $\Phi = \mathrm{char}_{\varpi^{-\mu}+\mathcal{O}}$*, then we have*

$$\zeta(2s - 1, \rho, \Phi) = \rho(\varpi^{-\mu})q^{(2s-2)\mu}(1 - q^{-1})^{-1}. \tag{4.2.8}$$

Proof. (i) is essentially done in [Tat67, p.319 - 322]. (ii) and (iii) can be checked by direct computations.□

Lemma 4.2.5. *For g defined at the beginning of this section, we have the* *following global expression for $f(g)$:*

$$f(g) = L_{S\cup\Sigma_\infty}(2s,\rho)Ny^s \prod_{v \text{ real}} L_{\mathbb{R}}(2s+|k_v|)e^{ik_v\theta_v} \prod_{v \text{ complex}} L_{\mathbb{C}}(2s+\lambda_v)y_v^{\lambda_v}\cdot$$
$$\prod_{v\in S_1} \xi_1(x_v)\rho_v(\varpi)^{-\mu_v} q_v^{(2\mu_v-m_v)s-\mu_v}(1-q_v^{-1})^{-1}\cdot$$
$$\prod_{v\in S_2} \xi_2(x_v)\rho_v(-1)q_v^{m_v s-\mu_v}(1-q_v^{-1})^{-1}. \tag{4.2.9}$$

Proof. The above formula is obtained by substituting local factors defined in sections 2.4.2 and 3.3.2. Here $L_{S\cup\Sigma_\infty}(s,\rho) = \prod_{v\notin S\cup\Sigma_\infty} L_v(s,\rho)$ is the product of L-factors over finite places which are unramified. □

Corollary 4.2.6. *When $s = 1$, we have:*

$$f(g) = L_{S\cup\Sigma_\infty}(2,\rho)Ny \prod_{v \text{ real}} L_{\mathbb{R}}(2+|k_v|)e^{ik_v\theta_v} \prod_{v \text{ complex}} L_{\mathbb{C}}(2+\lambda_v)y_v^{\lambda_v}\cdot$$
$$\prod_{v\in S_1} \xi_1(x_v)\rho_v(\varpi)^{-\mu_v} q_v^{\mu_v-m_v}(1-q_v^{-1})^{-1} \prod_{v\in S_2} \xi_2(x_v)\rho_v(-1)q_v^{m_v-\mu_v}(1-q_v^{-1})^{-1}. \tag{4.2.10}$$

Laurent expansions for $\mathcal{M}f(g)$

First we compute **local expressions** for $\mathcal{M}f(g)$.

Lemma 4.2.7. *At a real place, we have:*

$$\frac{\mathcal{M}f(g)}{\zeta(2s-1,\rho,\Phi_k)} = e^{ik\theta}y^{1-s}\pi^{-\frac{1}{2}}\frac{\Gamma(s+\frac{|k|}{2})}{\Gamma(s+\frac{|k|}{2}-\frac{1}{2})}I(s,k,0). \tag{4.2.11}$$

where $I(s,k,0)$ is given by (3.3.35).

Proof. By Lemma 2.4.9,

$$\mathcal{M}f(g) = e^{ik\theta}y^{1-s}\pi^{-(s+\frac{|k|}{2})}\Gamma(s+\frac{|k|}{2})\int_{\mathbb{R}}(1+u^2)^{-(s+\frac{k}{2})}(u-i)^k du \tag{4.2.12}$$

Recall from above that local L-factor is set to be :

$$\zeta(2s-1,\rho,\Phi_k)=\pi^{-(s+\frac{|k|}{2}-\frac{1}{2})}\Gamma(s+\frac{|k|}{2}-\frac{1}{2}). \tag{4.2.13}$$

Then take quotient.□

Lemma 4.2.8. *At a complex place, we have:*

$$\frac{\mathcal{M}f(g)}{\zeta(2s-1,\rho,\Phi)}=y^{2-2s-\lambda}. \tag{4.2.14}$$

Proof. At a complex place, we have from Lemma 2.4.11

$$\mathcal{M}f(g)=\pi^{-(2s+\lambda-2)}\Gamma(2s+\lambda-1)y^{2-2s-\lambda}, \tag{4.2.15}$$

and from Lemma 4.2.3

$$\zeta(2s-1,\rho,\Phi)=\pi^{-(2s+\lambda-2)}\Gamma(2s+\lambda-1).\square \tag{4.2.16}$$

Now putting them together we get the following expression over the archimedean places:

$$\begin{aligned}&\prod_{v\in\Sigma_\infty}\frac{\mathcal{M}f(g)}{\zeta_v(2s-1,\rho,\Phi_k)}\\&=\pi^{-\frac{r_1}{2}}Ny^{1-s}\prod_{v\text{ complex}}y^{-\lambda_v}\prod_{v\text{ real}}e^{ik_v\theta_v}\frac{\Gamma(s+\frac{|k_v|}{2})}{\Gamma(s+\frac{|k_v|}{2}-\frac{1}{2})}I(s,k_v,0).\end{aligned} \tag{4.2.17}$$

The following lemma deals with nonarchimedean places:

Lemma 4.2.9. *Suppose $S\neq\emptyset$.*

(i) If $v\in S_1$, then from (3.3.21)*, we have*

$$\mathcal{M}f(g)=\xi_2(x)\rho(-1)q^{(s-1)m-\mu}(1-q^{-1})^{-1}. \tag{4.2.18}$$

As a result of (4.2.7)*, the quotient is*

$$\frac{\mathcal{M}f(g)}{\zeta(2s-1,\rho,\Phi)} = \xi_2(x)\rho(-1)q^{(s-1)m}. \tag{4.2.19}$$

(ii) If $v \in S_2$, then from (3.3.22)*, we know*

$$\mathcal{M}f\Big(\begin{pmatrix} x & \\ & 1 \end{pmatrix} w_0\Big) = \xi_2(x)\rho(-\varpi^{-\mu})q^{(2\mu+m)(s-1)}(1-q^{-1})^{-1}. \tag{4.2.20}$$

As a result of (4.2.8)*, the quotient is:*

$$\frac{\mathcal{M}f(g)}{\zeta(2s-1,\rho,\Phi)} = \xi_2(x)\rho(-1)q^{(s-1)m}. \tag{4.2.21}$$

Now we compute $\mathcal{M}f$ for the **global intertwining operator $\mathcal{M}$**. Recall in (4.1.13) we have obtained the following expression for $\mathcal{M}f$:

$$\mathcal{M}f(g) = L(2s-1,\rho)\prod_{v\in S\cup\Sigma_\infty}\frac{\mathcal{M}_v f_v}{L_v(2s-1,\rho)}\Big(\frac{1}{L_{S\cup\Sigma_\infty}(2s,\rho)}\prod_{v\notin S\cup\Sigma_\infty} f_v^0\Big). \tag{4.2.22}$$

We will deduce propositions 4.2.10 and 4.2.11 by evaluating the above expressions at $s=1$.

For simplicity of expressions, we define the following function:

$$\begin{aligned} T(g) :=& \pi^{-\frac{r_1}{2}}\prod_{v\text{ complex}} y^{-\lambda_v}\prod_{v\text{ real}} e^{ik_v\theta_v}\frac{\Gamma(1+\frac{|k_v|}{2})}{\Gamma(1+\frac{|k_v|}{2}-\frac{1}{2})}I(1,k_v,0) \\ & L^{-1}_{S\cup\Sigma_\infty}(2,\rho)\prod_{v\notin S\cup\Sigma_\infty}\tilde{f}_v^0(g,1). \end{aligned} \tag{4.2.23}$$

where $\tilde{f}_v^0(g,1)$ is obtained from evaluating $\tilde{f}_v^0$ at $s=1$.

S **is empty.** Now let us assume that $\rho = 1$ so that S is empty. From Proposition 4.1.7, we know that our Eisenstein series has a simple pole at $s=1$. Now we can write down the Laurent expansion of $\mathcal{M}f$ based on the above analysis of local factors.

Denote by R and C_0 the residue and constant term of the Laurent expansion of $L(s,\rho)$ at $s=1$ respectively.

Proposition 4.2.10. *When S is empty, the Laurent expansion of $\mathcal{M}f(g)$ at $s=1$ is:*

$$\frac{T(g)R}{s-1} + T(g)C_0 - T(g)R\log Ny + O(s-1). \tag{4.2.24}$$

Proof. Using the above expression of $\mathcal{M}f$, we get:

$$\begin{aligned}\mathcal{M}f(g) &= \Big(\frac{R}{s-1} + C_0 + O(s-1)\Big)\cdot\Big(1-(\log Ny)(s-1) + O(s-1)^2\Big)T(g)\\ &= \frac{T(g)R}{s-1} + T(g)C_0 - T(g)R\log Ny + O(s-1).\end{aligned} \tag{4.2.25}$$

□

S **is nonempty.** In this case, we know that E is holomorphic. We can simply evaluate E at $s=1$ to get a 'limit formula'. First we can evaluate $\mathcal{M}f(g)$ at $s=1$ and obtain the following expression.

Proposition 4.2.11. *When S is nonempty, we have the following expression for $\mathcal{M}f$ at $s=1$:*

$$T(g)L(1,\rho)\prod_{v\in S}\xi_2(x_v)\rho_v(-1). \tag{4.2.26}$$

4.2.1 Limit formula when E has a pole

First we compute the Whittaker functions in the Fourier expansion of E based on chapter 3.

Lemma 4.2.12. *For g given at the beginning of section 4.2, the Fourier* *coefficients of E have the following expression at $s=1$:*

$$\begin{aligned}&\sum_{\epsilon\in F^\times} W\left(\begin{pmatrix}\epsilon & \\ & 1\end{pmatrix} g\right)\Big|_{s=1}\\ &= (2\pi)^{r_2}\sum_{\epsilon\in F^\times}\bar{\Psi}(\epsilon x)\prod_{v\ real}\rho_v(-1)e^{ik_v\theta_v}L_{\mathbb{R}}(2+|k_v|)I(1,k_v,y_v\epsilon)\cdot\\ &\prod_{v\ complex}(y_v\operatorname{Re}\epsilon)^{\lambda_v+1}K_{\lambda_v+1}(4\pi y_v\operatorname{Re}\epsilon)\prod_{v\nmid\infty}\xi_{2,v}(\det g)\sum_{0\le v(\tau)\le m_v+n_v}\rho_v(\tau)|\tau|_v\end{aligned}\tag{4.2.27}$$

where $n_v = v(\epsilon)$ and $\tau\in\mathcal{O}_v$.

Proof. We just need to put together our computations done in section 3.3.

Real places. From Proposition 3.3.9 and under the same setting, we have for $s=1$:

$$\begin{aligned}&W\left(\begin{pmatrix}\epsilon & \\ & 1\end{pmatrix}\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}y^{\frac12} & \\ & y^{-\frac12}\end{pmatrix}k_\theta\right)\\ &= e^{ik\theta}\rho(-\epsilon^{-1})\xi_1(\epsilon)\bar{\psi}(\epsilon x)L_{\mathbb{R}}(2+|k|)I(1,k,y\epsilon)\end{aligned}\tag{4.2.28}$$

Complex places. From Proposition 3.3.11, we can evaluate the expression at $s=1$:

$$\begin{aligned}&W\left(\begin{pmatrix}\epsilon & \\ & 1\end{pmatrix}\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}y^{\frac12} & \\ & y^{-\frac12}\end{pmatrix}k\right)\\ &= \xi_1(\epsilon)\rho(-\epsilon^{-1})(2\pi)(y\operatorname{Re}\epsilon)^{\lambda+1}\bar{\psi}(\epsilon x)K_{\lambda+1}(4\pi y\operatorname{Re}\epsilon)\end{aligned}\tag{4.2.29}$$

with $\operatorname{Re}\lambda > -\frac{7}{4}$.

Nonarchimedean places. First we rewrite the formula for spherical Whittaker functions in Corollary 3.3.2 so that it looks more transparent when we put them together to get a global expression. For the moment as we are working locally, we omit the subscript v.

According to the definition at the beginning of section 4.2, $g = \begin{pmatrix} \varpi^m & \\ & 1 \end{pmatrix}$.

For $\epsilon \in F^\times$, we denote its valuation $v(\epsilon)$ by n. We can rewrite $W\left(\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix} g\right)$ in the following way (with same assumptions on χ_1, χ_2 as in Corollary 3.3.2):

$$W\left(\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix} g\right) = q^{-\frac{m+n}{2}} \frac{\alpha_1^{m+n+1} - \alpha_2^{m+n+1}}{\alpha_1 - \alpha_2}. \tag{4.2.30}$$

Notice that $\alpha_1\alpha_2^{-1} = q^{1-2s}\rho(\varpi)$. We can write (4.2.30) in terms of ϵ and g:

$$\begin{aligned} &q^{-\frac{m+n}{2}} \alpha_2^{m+n}\left(\left(\frac{\alpha_1}{\alpha_2}\right)^{m+n} + \left(\frac{\alpha_1}{\alpha_2}\right)^{m+n-1} + \ldots + 1\right) \\ &= |\epsilon \det g|^{1-s}\xi_2(\epsilon \det g) \sum_{0 \le v(\tau) \le m+n} |\tau|^{2s-1}\rho(\tau), \quad \tau \in \mathcal{O}. \end{aligned} \tag{4.2.31}$$

Taking product over all nonarchimedean places, we get:

$$\begin{aligned} &\prod_{v \nmid \infty} W_v\left(\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix} g_v\right) \\ &= N|\epsilon \det g|^{s-1} \prod_{v \nmid \infty} \xi_{2,v}(\epsilon \det g) \sum_{0 \le v(\tau) \le m_v+n_v} |\tau|_v^{2s-1}\rho_v(\tau), \quad \tau \in \mathcal{O}_v. \end{aligned} \tag{4.2.32}$$

Evaluating the expression at $s = 1$, we get:

$$\prod_{v \nmid \infty} W_v\left(\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix} g_v\right)\Big|_{s=1} = \prod_{v \nmid \infty} \xi_{2,v}(\epsilon \det g) \sum_{0 \le v(\tau) \le m_v+n_v} \rho_v(\tau)|\tau|_v, \quad \tau \in \mathcal{O}_v. \tag{4.2.33}$$

Putting together archimedean and nonarchimedean components and notice that ξ is trivial on F and $\rho_v(-1) = 1$ when v is complex, we get formula (4.2.27). $\square$

Theorem 4.2.13 (Kronecker limit formula when E has a pole)**.**

$$\begin{aligned}
&\lim_{s\to 1}\left\{E(g,s,\xi_1,\xi_2,\phi)-\frac{T(g)R}{s-1}\right\}\\
&=T(g)C_0+L_{\Sigma_\infty}(2,\rho)Ny\prod_{v\ real}L_{\mathbb{R}}(2+|k_v|)e^{ik_v\theta_v}\prod_{v\ complex}L_{\mathbb{C}}(2+\lambda_v)y_v^{\lambda_v}\\
&-T(g)R\log Ny+\\
&(2\pi)^{r_2}\sum_{\epsilon\in F^\times}\bar{\Psi}(\epsilon x)\prod_{v\ real}\rho_v(-1)e^{ik_v\theta_v}L_{\mathbb{R}}(2+|k_v|)I(1,k_v,y_v\epsilon)\cdot\\
&\prod_{v\ complex}(y_v\operatorname{Re}\epsilon)^{\lambda_v+1}K_{\lambda_v+1}(4\pi y_v\operatorname{Re}\epsilon)\prod_{v\nmid\infty}\xi_{2,v}(\det g)\sum_{0\le v(\tau)\le m_v+n_v}\rho_v(\tau)|\tau|_v,
\end{aligned}\tag{4.2.34}$$

where $\Psi(\epsilon x)=\prod_{v\in\Sigma_\infty}\psi_v(\epsilon x_v)$,

$$C_0=\lim_{s\to 1}\left\{L(s,\rho)-\frac{R}{s-1}\right\}\tag{4.2.35}$$

and $T(g)$ *is given by* (4.2.23).

The analogue of Asai's function is given by:

$$\begin{aligned}
h(g)=&\,L_{\Sigma_\infty}(2,\rho)Ny\prod_{v\ real}L_{\mathbb{R}}(2+|k_v|)e^{ik_v\theta_v}\prod_{v\ complex}L_{\mathbb{C}}(2+\lambda_v)y_v^{\lambda_v}\\
&+(2\pi)^{r_2}\sum_{\epsilon\in F^\times}\bar{\Psi}(\epsilon x)\prod_{v\ real}\rho_v(-1)e^{ik_v\theta_v}L_{\mathbb{R}}(2+|k_v|)I(1,k_v,y_v\epsilon)\cdot\\
&\prod_{v\ complex}(y_v\operatorname{Re}\epsilon)^{\lambda_v+1}K_{\lambda_v+1}(4\pi y_v\operatorname{Re}\epsilon)\prod_{v\nmid\infty}\xi_{2,v}(\det g)\sum_{0\le v(\tau)\le m_v+n_v}\rho_v(\tau)|\tau|_v.
\end{aligned}\tag{4.2.36}$$

The term $I(1,k_v,y_v\epsilon)$ *has the following expression:*

$$I(1,k_v,y_v\epsilon)=\begin{cases}\frac{2\pi}{k_v}\Gamma^{-1}\left(\frac{k_v}{2}\right)W_{\frac{k_v}{2},-\frac{1}{2}}(4\pi y_v\epsilon), & \epsilon>0,\\ -\frac{2\pi}{k_v}\Gamma^{-1}\left(-\frac{k_v}{2}\right)W_{-\frac{k_v}{2},-\frac{1}{2}}(-4\pi y_v\epsilon), & \epsilon<0.\end{cases}\tag{4.2.37}$$

Proof. Substituting the expression (4.2.25) for $\mathcal{M}f$ and the expression of $f(g)$ at $s=1$ in Corollary 4.2.6 to the Fourier expansion of $E(g,s,\xi_1,\xi_2,\phi)$

in (3.1.10), we get the following formula at $s=1$:

$$\begin{aligned}
&\lim_{s\to 1}\left\{E(g,s,\xi_1,\xi_2,\phi)-\frac{TR}{s-1}\right\}\\
&=T(g)C_0+L_{\Sigma_\infty}(2,\rho)Ny\prod_{v\text{ real}}L_{\mathbb{R}}(2+|k_v|)e^{ik_v\theta_v}\prod_{v\text{ complex}}L_{\mathbb{C}}(2+\lambda_v)y_v^{\lambda_v}\\
&-T(g)R\log Ny+\sum_{\epsilon\in F^\times}W_1\left(\begin{pmatrix}\epsilon & \\ & 1\end{pmatrix}g\right),
\end{aligned} \tag{4.2.38}$$

where $W_1\left(\begin{pmatrix}\epsilon & \\ & 1\end{pmatrix}g\right)$ is the Whittaker function $W\left(\begin{pmatrix}\epsilon & \\ & 1\end{pmatrix}g\right)$ evaluated at $s=1$.

Now use Lemma 4.2.12. The expression for $I(1,k_v,y_v\epsilon)$ is obtained using (3.3.52).□

Corollary 4.2.14. *When F is totally real of degree d and $k_v=0$ for all v real, we have the following expression for $h(g)$:*

$$\begin{aligned}
h(g)=&\pi^{-d}L_{\Sigma_\infty}(2,\rho)Ny+\sum_{\epsilon\in F^\times}\bar{\Psi}(\epsilon x)e^{-2\pi\sum\limits_{v|\infty}y_v|\epsilon|}\cdot\\
&\prod_{v\nmid\infty}\xi_{2,v}(\det g_v)\frac{1-(\rho_v(\varpi)q_v^{-1})^{m_v+n_v+1}}{1-\rho_v(\varpi)q_v^{-1}},
\end{aligned} \tag{4.2.39}$$

where $m_v=v(\epsilon)$.

Proof. It is worth mentioning how the integral I is simplified. Making use of (3.3.50), we have

$$I(1,0,y_v\epsilon)=2\pi|y_v\epsilon|^{\frac{1}{2}}K_{\frac{1}{2}}(2\pi|y_v\epsilon|). \tag{4.2.40}$$

Recall the formula for $K_{\frac{1}{2}}$ in (1.2.11), we get

$$I(1,0,y_v\epsilon)=\pi e^{-2\pi y_v|\epsilon|},y_v>0.\square \tag{4.2.41}$$

4.2.2 Limit formula when E is holomorphic

We know that when ρ is ramified, E is an entire function of s. Using the Fourier expansion for Eisenstein series again, we can get a second limit formula at $s = 1$.

Theorem 4.2.15 (Kronecker limit formula when E is holomorphic)**.**
When S is nonempty, $E(g, s, \xi_1, \xi_2, \phi)$ is holomorphic at $s = 1$. For g given by the beginning of section 4.2, the following is valid:

$$E(g, 1, \xi_1, \xi_2, \phi) = f(g, 1, \xi_1, \xi_2, \phi) + \mathcal{M}f(g, 1, \xi_1, \xi_2, \phi) + \sum_{\epsilon \in F^\times} W\left(\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix} g \right)\Big|_{s=1}, \tag{4.2.42}$$

where

$$\begin{aligned} &f(g, 1, \xi_1, \xi_2, \phi) \\ &= L_{S\cup\Sigma_\infty}(2, \rho) N y \prod_{v \text{ real}} L_{\mathbb{R}}(2 + |k_v|) e^{i k_v \theta_v} \prod_{v \text{ complex}} L_{\mathbb{C}}(2 + \lambda_v) y_v^{\lambda_v} \cdot \\ &\prod_{v \in S_1} \xi_{1,v}(\epsilon \det g) \rho_v(\varpi)^{-\mu_v} q_v^{\mu_v - m_v} (1 - q_v^{-1})^{-1} \prod_{v \in S_2} \xi_{1,v}(\epsilon \det g) \rho_v(-1) q_v^{-m_v - \mu_v} (1 - q_v^{-1})^{-1}, \end{aligned} \tag{4.2.43}$$

$$\mathcal{M}f(g, 1, \xi_1, \xi_2, \phi) = T(g) L(1, \rho) \prod_{v \in S} \xi_{2,v}(\epsilon \det g) \rho_v(-1). \tag{4.2.44}$$

and

$$\sum_{\epsilon\in F^\times} W\Big(\begin{pmatrix}\epsilon & \\ & 1\end{pmatrix} g\Big)\Big|_{s=1}$$
$$= (2\pi)^{r_2} \sum_{\epsilon\in F^\times} \bar{\Psi}(\epsilon x) \prod_{v\ real} \rho_v(-1) e^{ik_v\theta_v} L_{\mathbb{R}}(2+|k_v|) I(1,k_v,y_v\epsilon)\cdot$$
$$\prod_{v\ complex} (y \operatorname{Re}\epsilon)^{\lambda_v+1} K_{\lambda_v+1}(4\pi y_v \operatorname{Re}\epsilon) \prod_{v\notin S\cup\Sigma_\infty} \xi_{2,v}(\det g) \sum_{0\le v(\tau)\le m_v+n_v} \rho_v(\tau)|\tau|_v$$
$$\prod_{v\in S_1} \xi_{2,v}(\det g) q_v^{m_v+n_v-\mu_v}(1-q_v^{-1})^{-1}\cdot$$
$$\prod_{v\in S_2} \rho_v(-\epsilon\varpi^{-\mu_v})\xi_{2,v}(\det g)(1-q_v^{-1})^{-1} q_v^{-\max\{0,-m_v-n_v-\mu_v-l_v\}}, \tag{4.2.45}$$

where $\operatorname{Re}(s) > \frac{1}{8}$ *and* $I(1,k_v,y_v\epsilon)$ *can be expressed by* (4.2.37).

Proof. The formulae for f and $\mathcal{M}f$ are from (4.2.10) and (4.2.25) respectively. The expression for the higher Fourier coefficients is obtained from (3.3.34), (3.3.53), (4.2.31), (3.3.25) and (3.3.26). Notice that we have made use of the fact that the global character $\rho = \xi_1\xi_2^{-1}$ is trivial when restricted to the number field.□

Corollary 4.2.16. *When F is totally real of degree d and $k_v = 0$ for all* *archimedean places v, we have the following expression for E:*

$$
\begin{aligned}
&E(g,1,\xi_1,\xi_2,\phi) \\
&= L_{S\cup\Sigma_\infty}(2,\rho)Ny\prod_{v|\infty}L_{\mathbb{R}}(2+|k_v|)e^{ik_v\theta_v}\cdot \\
&\prod_{v\in S_1}\xi_{1,v}(\epsilon\det g)\rho_v(\varpi)^{-\mu_v}q_v^{\mu_v-m_v}(1-q_v^{-1})^{-1}\prod_{v\in S_2}\xi_{1,v}(\epsilon\det g)\rho_v(-1)q_v^{-m_v-\mu_v}(1-q_v^{-1})^{-1}+ \\
&T(g)L(1,\rho)\prod_{v\in S}\xi_{2,v}(\epsilon\det g)\rho_v(-1)+ \\
&\sum_{\epsilon\in F^\times}\bar{\Psi}(\epsilon x)\prod_{v|\infty}e^{ik_v\theta_v}L_{\mathbb{R}}(2+|k_v|)I(1,k_v,y_v\epsilon)\cdot \\
&\prod_{v\notin S\cup\Sigma_\infty}\xi_{2,v}(\det g)\sum_{0\le v(\tau)\le m_v+n_v}\rho_v(\tau)|\tau|_v \\
&\prod_{v\in S_1}\xi_{2,v}(\det g)q_v^{m_v+n_v-\mu_v}(1-q_v^{-1})^{-1}\cdot \\
&\prod_{v\in S_2}\rho_v(-\epsilon\varpi^{-\mu_v})\xi_{2,v}(\det g)(1-q_v^{-1})^{-1}q_v^{-\max\{0,-m_v-n_v-\mu_v-l_v\}}.
\end{aligned}
\tag{4.2.46}
$$

where $I(1,k_v,y_v\epsilon)$ can be expressed by (4.2.37).

Chapter 5

The Rankin-Selberg integral

In this chapter, we make use of adelic Eisenstein series and our knowledge of their Fourier expansions to generalise Scholl's work on Rankin-Selberg method in [Sch98, Chap 4].

Notations:
F: Either a number field or local field, as the beginning of Chapter 2.
ψ will be a character on $\mathbb{A}/F$ whose complex conjugate is given in section 2.3.

5.1 Global and local Rankin-Selberg integrals

We know that Weil's converse theorem [Miy89, Thm 4.3.15] tells us that if an L-function has analytic continuation and has sufficiently many functional equations, then it is an L-function attached to a modular form. Rankin-Selberg method can be thought as a tool to represent L-functions as integrals against Eisenstein series. Using our knowledge of analytic continuations and functional equations of Eisenstein series, we could deduce properties for the L-function.

Rankin-Selberg integrals representing L-functions occur in many places. An explicit calculation of Rankin-Selberg integrals is an important ingredient in Kato's work on BSD conjecture (ref. [Kat04]). In [Sch98, Chap 4], Scholl gave an adelic formulation of calculations relevant to Kato's work. He computed explicitly the Rankin-Selberg integral of a cusp form against two Eisenstein series over the ground field $\mathbb{Q}$. Here we are interested in generalising his work to an arbitrary number field, based on Jacquet's work on Rankin-Selberg method on $\mathrm{GL}_2 \times \mathrm{GL}_2$. According to the philosophy of "plectic cohomology" in [NS15], these computations should be useful in generalising Kato's work to totally real fields.

Notations and settings Let φ be a cusp form generating an irreducible cuspidal representation π with central character ω.

Let $\pi_1 = \mathcal{B}(\chi_1, \chi_2), \pi_2 = \mathcal{B}(\Theta_1, \Theta_2)$ be admissible representations induced from Hecke characters $\chi_i, \Theta_i \colon \mathbb{A}^\times / F^\times \to \mathbb{C}^\times, i = 1, 2$. Take Eisenstein series E_1, E_2 which are constructed from sections in $\mathcal{B}(\chi_1, \chi_2)$ and $\mathcal{B}(\Theta_1, \Theta_2)$ respectively, as in section 2.4.1.

Put

$$\chi_1 = |\ |^{s_1 - \frac{1}{2}} \xi_1, \chi_2 = |\ |^{\frac{1}{2} - s_1} \xi_2, \Theta_1 = |\ |^{s_2 - \frac{1}{2}} \eta_1, \Theta_2 = |\ |^{\frac{1}{2} - s_2} \eta_2 \tag{5.1.1}$$

where $s_1, s_2 \in \mathbb{C}$. Write $\sigma = \eta_1 \eta_2^{-1}$ and $\rho = \xi_1 \xi_2^{-1}$.

It is more convenient to assume that $\sigma = \eta_1$ and $\eta_2 = 1$ which in fact does not lose generality since we can always twist the representation π_2 by η_2^{-1}, i.e. replace π_2 by $\pi_2 \otimes \eta_2^{-1}$. Similarly, we can assume $\rho = \xi_1$. Notice that under these assumptions, the central character of E_1 (E_2 resp.) is ρ (σ resp.).

To make the function $\Omega = E_1 E_2 \varphi$ invariant under the centre $Z(\mathbb{A})$ of $G(\mathbb{A})$, we assume:

$$\omega^{-1} = \rho \sigma. \tag{5.1.2}$$

Our goal in section 5.1 is to compute the integral:

$$\int_{G(F) Z(\mathbb{A}) \backslash G(\mathbb{A})} \Omega \, dg. \tag{5.1.3}$$

Remark 5.1.1. The case when Ω is a product of one Eisenstein series and two cusp forms is done in [Jac72, Prop 15.2]. The present case, at least at good primes, behaves very similarly.

Global and local Rankin-Selberg integrals

First we show that the goal can be achieved by computing the local components of the Rankin-Selberg integral, i.e. Proposition 5.1.2.

Suppose φ, E_2 have the following Fourier expansions:

$$\begin{aligned} \varphi(g) &= \sum_{\epsilon \in F^\times} W\left(\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix} g\right), \\ E_2(g) &= f_2(g) + \mathcal{M} f_2(g) + \sum_{\epsilon \in F^\times} W_2\left(\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix} g\right). \end{aligned} \tag{5.1.4}$$

Furthermore, write

$$E_1(g) = \sum_{B(F) \backslash G(F)} f_1(\gamma g) \tag{5.1.5}$$

where $f_1 \in \mathcal{B}(\chi_1, \chi_2)$.

For any place v, consider the following local Rankin-Selberg integral:

$$\mathcal{Z}_v(s_1, W, W_2, \rho) := \int_{F_v^\times \times G(\mathcal{O}_v)} f_1 W_2 W\left(\begin{pmatrix} t & \\ & 1 \end{pmatrix} h\right) d^\times t dh. \tag{5.1.6}$$

Since the integrand is invariant under the center $Z(\mathbb{A})$ of $\mathrm{GL}(\mathbb{A})$ by construction, we can replace $G(\mathcal{O}_v)$ by $\mathrm{SL}_2(\mathcal{O}_v)$ in the definition of $\mathcal{Z}_v(s, W, W_2, f_1)$,

$$\mathcal{Z}_v(s_1, W, W_2, \rho) = \int_{F_v^\times \times \mathrm{SL}_2(\mathcal{O}_v)} f_1 W_2 W\left(\begin{pmatrix} t & \\ & 1 \end{pmatrix} h\right) d^\times t dh. \tag{5.1.7}$$

Also notice that both $\mathrm{SL}_2(\mathcal{O}_v)$ and $G(\mathcal{O}_v)$ have the same measure which is one since the centre $\mathcal{O}_v^\times$ has measure one.

The following proposition shows that the Rankin-Selberg integral in (5.1.3) can be expressed as a product of local integrals (5.1.7) (ref [Bum97, Prop 3.8.2]).

Proposition 5.1.2. *With the terms defined above, the following formula is valid:*

$$\int_{G(F)Z(\mathbb{A})\backslash G(\mathbb{A})} \Omega dg = \prod_{v\in\Sigma} \mathcal{Z}_v(s_1, W, W_2, \rho), \tag{5.1.8}$$

with $\mathcal{Z}_v(s_1, W, W_2, \rho)$ *given in* (5.1.7) *and the product is taken over all places.*

Remark 5.1.3. Notice that our expression is different from that of [Bum97, Prop. 3.8.2] because the choice of our section f has L-factors that are built in in the definition.

Proof. First we can unravel the definition of E_1 and make use of the $G(F)$ invariance of an automorphic form to get the following:

$$\begin{aligned}\int_{G(F)Z(\mathbb{A})\backslash G(\mathbb{A})} E_1(g)E_2(g)\varphi(g)dg &= \int_{G(F)Z(\mathbb{A})\backslash G(\mathbb{A})} \sum_{B(F)\backslash G(F)} f_1(\gamma g)E_2(g)\varphi(g)dg \\ &= \int_{B(F)Z(\mathbb{A})\backslash G(\mathbb{A})} f_1(g)E_2(g)\varphi(g)dg.\end{aligned} \tag{5.1.9}$$

Now notice that $G(\mathbb{A}) = B(\mathbb{A})\,\mathrm{SL}_2(\widehat{\mathcal{O}})$ and we can write the parabolic $B(\mathbb{A})$ as the following product involving the unipotent matrix $N(\mathbb{A})$:

$$B(\mathbb{A}) = Z(\mathbb{A})T_1(\mathbb{A})N(\mathbb{A}), \tag{5.1.10}$$

where $T_1(\mathbb{A})$ is the subgroup of the maximal torus $T(\mathbb{A})$ with lower diagonal entry 1. Similar decomposition applies to $B(F)$. As a result

$$B(F)Z(\mathbb{A})\backslash G(\mathbb{A}) \cong \mathrm{SL}_2(\widehat{\mathcal{O}}) \times [T_1(F)\backslash T_1(\mathbb{A})] \times [N(F)\backslash N(\mathbb{A})]. \tag{5.1.11}$$

As a result, we have a decomposition for the Haar measure dg:

$$dg = du \cdot dn \cdot dt \cdot dk, \tag{5.1.12}$$

where du, dn, dt, dk are measures on $Z(\mathbb{A})$, $N(F)\backslash N(\mathbb{A}), T_1(F)\backslash T_1(\mathbb{A})$ and $\mathrm{SL}_2(\widehat{\mathcal{O}})$ respectively.

Now observe that

$$\begin{aligned}
&\int_{B(F)Z(\mathbb{A})\backslash G(\mathbb{A})} f_1(g)E_2(g)\varphi(g)dg \\
&= \int_{B(F)Z(\mathbb{A})\backslash G(\mathbb{A})} f_1(g)\sum_{\epsilon\in F^\times} W_2\left(\begin{pmatrix}\epsilon & \\ & 1\end{pmatrix} g\right)\varphi(g)dg,
\end{aligned} \tag{5.1.13}$$

because the other term involving the constant term of $E_2(g)$ vanishes as φ has zero constant term.

From the decomposition in (5.1.11), we have:

$$\begin{aligned}
&\int_{B(F)Z(\mathbb{A})\backslash G(\mathbb{A})} f_1(g)\sum_{\epsilon\in F^\times} W_2\left(\begin{pmatrix}\epsilon & \\ & 1\end{pmatrix} g\right)\varphi(g)dg \\
&= \int_{\mathrm{SL}_2(\widehat{\mathcal{O}})\times[T_1(F)\backslash T_1(\mathbb{A})]\times[N(F)\backslash N(\mathbb{A})]} f_1(tnk)\sum_{\epsilon\in F^\times} W_2\left(\begin{pmatrix}\epsilon & \\ & 1\end{pmatrix} tnk\right)\varphi(tnk)dk\cdot dt\cdot dn \\
&= \int_{\mathrm{SL}_2(\widehat{\mathcal{O}})\times T_1(\mathbb{A})\times[N(F)\backslash N(\mathbb{A})]} f_1(tnk)W_2(tnk)\varphi(tnk)dk\cdot dt\cdot dn.
\end{aligned} \tag{5.1.14}$$

Next we show that we can absorb the integration over $N(F)\backslash N(\mathbb{A})$ into the integrand. First notice that we can perform a change of variable:

$$n\mapsto t^{-1}nt, \tag{5.1.15}$$

and the measure dn becomes $\delta_B^{-1}(t)dn$, where δ_B is the modulus character on B. Then since f_1 is $N(\mathbb{A})$-invariant, we can write

$$\begin{aligned}
&\int_{\mathrm{SL}_2(\widehat{\mathcal{O}})\times T_1(\mathbb{A})\times[N(F)\backslash N(\mathbb{A})]} f_1(tnk)W_2(tnk)\varphi(tnk)dk\cdot dt\cdot dn \\
&= \int_{\mathrm{SL}_2(\widehat{\mathcal{O}})\times T_1(\mathbb{A})\times[N(F)\backslash N(\mathbb{A})]} f_1(tk)W_2(ntk)\varphi(ntk)\delta_B^{-1}(t)dk\cdot dt\cdot dn \\
&= \int_{\mathrm{SL}_2(\widehat{\mathcal{O}})\times T_1(\mathbb{A})} f_1(tk)W_2(tk)\int_{N(F)\backslash N(\mathbb{A})}\psi(n)\varphi(ntk)dn\delta_B^{-1}(t)dk\cdot dt.
\end{aligned} \tag{5.1.16}$$

Now we recognise that the integral over the unipotent part is just the Fourier coefficient W of φ.

So the last line is:

$$\int_{\mathrm{SL}_2(\widehat{\mathcal{O}})\times T_1(\mathbb{A})} f_1(tk)W_2(tk)W(tk)\delta_B^{-1}(t)dk \cdot dt, \qquad (5.1.17)$$

which is the right hand side of (5.1.8). Notice that we can regard t as the upper entry of a matrix in $T_1(\mathbb{A})$ and the measure $\delta_B^{-1}(t)dt$ is the same as the multiplicative measure $d^\times t$ on $\mathbb{A}^\times$. $\square$

Further assumptions We now compute $\mathcal{Z}_v$ for all places v. Denote the degree of F by d. We assume that the weights of E_1, E_2 and φ are k, l, and $-k-l$ (which are multiple indices) respectively, where $k_i, l_i \in \mathbb{Z}, 1 \leq i \leq d$. Recall that an automorphic form $\tilde{\varphi}$ is said to have weight $k = (k_1, ..., k_d)$ if for any $g \in G(\mathbb{A})$:

$$\tilde{\varphi}(gh(\theta_j)) = e^{ik_j\theta_j}\tilde{\varphi}(g), \qquad (5.1.18)$$

for any $h(\theta_j) \in SO_2, 1 \leq j \leq d$.

There are two cases to consider for the representation π at a nonarchimedean place v:

(i) v **unramified or good**, meaning $\pi_v, \pi_{1,v}, \pi_{2,v}$ are all spherical principal series. Assume that the conductor of ψ is $\mathcal{O}_v$ for all unramified places v.

(ii) v **bad**, meaning π_v is supercuspidal.

It might also be possible to treat other cases such as ramified principal series and special representations in explicit ways similar to ours here, based on results of Kirillov models of those cases (see for example [God70, p 1.36]). However, we will not carry out this discussion in the thesis.

We **assume that ρ is unramified at each nonarchimedean place**.

5.1.1 Unramified nonarchimedean place

In this and the following sections, **we work locally at a fixed place** v, and write π to denote π_v.

Denote the Satake parameters of π_1 and π by α, γ respectively. Under the notation at the beginning of section 5.1, this means $\alpha_i = \chi_i(\varpi), i = 1, 2$. We define γ in a similar way without writing down corresponding characters.

Proposition 5.1.4. *At an unramified place with $\chi_i, \Theta_i, i = 1, 2$ satisfying* (5.1.1) *and* (5.1.2)*, the following formula is valid:*

$$\begin{aligned}\mathcal{Z}(s_1, W, W_2, \rho) =& L(2s_1, \rho)L(2s_1 + 2, \rho)^{-1} L\left(\pi \otimes \bar{\omega}, s_1 + s_2 + \frac{1}{2}\right) \cdot \\ & L\left(\pi \otimes \rho, s_1 - s_2 + \frac{3}{2}\right).\end{aligned} \tag{5.1.19}$$

Proof. Since f_1 is spherical, we know by Proposition 2.4.3 that for $t \in F^\times$:

$$f_1 \begin{pmatrix} t & \\ & 1 \end{pmatrix} = \chi_1(t)|t|^{\frac{1}{2}} f(1) = \rho(t)|t|^{s_1}(1 - \rho(\varpi)q^{-2s_1})^{-1}. \tag{5.1.20}$$

Now substituting the above expression of f_1 into (5.1.6), we get

$$\begin{aligned}\mathcal{Z}(s_1, W, W_2, \rho) &= L(2s_1, \rho) \int_{\mathcal{O}\setminus\{0\}} \rho(t)|t|^{s_1} W_2 W \begin{pmatrix} t & \\ & 1 \end{pmatrix} d^\times t \\ &= L(2s_1, \rho) \sum_{n=0}^{\infty} (\rho(\varpi)q^{-s_1})^n W_2 W \begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix}.\end{aligned} \tag{5.1.21}$$

To simplify the sum we make use of the following lemma:

Lemma 5.1.5 ([Jac72, Lem 15.9.4])**.** *If*

$$\begin{aligned}\sum_{r=0}^{\infty} A(r)x^r &= (1 - a_1 x)^{-1}(1 - a_2 x)^{-1}, \\ \sum_{r=0}^{\infty} B(r)x^r &= (1 - b_1 x)^{-1}(1 - b_2 x)^{-1},\end{aligned} \tag{5.1.22}$$

then

$$\sum_{r=0}^{\infty} A(r)B(r)x^r = (1 - a_1a_2b_1b_2x^2)\prod_{i=1}^{2}\prod_{j=1}^{2}(1 - a_ib_jx)^{-1}. \tag{5.1.23}$$

Since W, W_2 satisfy Proposition 3.3.1, we have for γ_i given at the beginning of this section:

$$\begin{aligned}
\sum_{n=0}^{\infty} W\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} x^n &= (1 - \gamma_1 q^{-\frac{1}{2}}x)^{-1}(1 - \gamma_2 q^{-\frac{1}{2}}x)^{-1} \\
\sum_{n=0}^{\infty} W_2\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} x^n &= (1 - \sigma(\varpi)q^{-s_2}x)^{-1}(1 - q^{s_2-1}x)^{-1}.
\end{aligned} \tag{5.1.24}$$

This can be proved by writing the right hand sides in geometric series and comparing with formulae for W (and W_2) in Proposition 3.3.1.

Now applying the lemma to W, W_2 and $x = \rho(\varpi)q^{-s_1}$ with assumptions (5.1.1) and (5.1.2), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}(\rho(\varpi)q^{-s_1})^n W_2 W\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix} \\
&= (1 - \rho(\varpi)q^{-2(s_1+1)})(1 - \gamma_1\rho(\varpi)q^{-(s_1-s_2+\frac{3}{2})})^{-1}(1 - \gamma_2\rho(\varpi)q^{-(s_1-s_2+\frac{3}{2})})^{-1}\cdot \\
&\quad (1 - \gamma_1\omega(\varpi)q^{-(s_1+s_2+\frac{1}{2})})^{-1}(1 - \gamma_2\omega(\varpi)q^{-(s_1+s_2+\frac{1}{2})})^{-1} \\
&= L(2s_1 + 2, \rho)^{-1} L\left(\pi \otimes \bar{\omega}, s_1 + s_2 + \frac{1}{2}\right) L\left(\pi \otimes \rho, s_1 - s_2 + \frac{3}{2}\right).
\end{aligned} \tag{5.1.25}$$

Substituting (5.1.25) to (5.1.21), we get (5.1.19).□

5.1.2 Bad primes

Now we compute the Rankin-Selberg integral at a bad prime. Observe that we cannot apply the same technique in computing the Whittaker function at this place as that is applicable only for principal series representations which is not the case for π now.

The idea from [Sch98] is to make suitable choices of Eisenstein series E_1

and E_2 so that at a bad prime, the integrand Ω is right invariant under $K_1(\mathfrak{p}^\mu)$ for some μ and the Kirillov function associated to E_2 is essentially the characteristic function on local units. This will enable us to compute the integral without computing the Whittaker function associated to φ. We now explain how this can be done in our more general setting.

Construction of the section f_1 of E_1 The goal of constructing f_1 and W_2 in the following way is to make the Rankin-Selberg integral right invariant under $K_1(\mathfrak{p}^{\mu+1})$ and nonzero on a subset of

$$\begin{pmatrix} \mathcal{O}^\times & \\ & 1 \end{pmatrix} \cdot K_0(\mathfrak{p}^{\mu+1}),$$

for some $\mu \geq 0$.

To construct f_1, we follow Proposition 2.4.4 and take $\phi = \mathrm{char}_{\mathfrak{p}^{\mu+1} \times \mathcal{O}^\times}$. We know from the same proposition that for $g \in G(\mathcal{O})$,

$$f_1(g) \neq 0 \Leftrightarrow g \in K_0(\mathfrak{p}^{\mu+1}). \tag{5.1.26}$$

And if ρ is unramified, then for any $x \in F^\times$,

$$f_1 \begin{pmatrix} x & \\ & 1 \end{pmatrix} = |x|^{s_1} \rho(x). \tag{5.1.27}$$

Construction of the Kirillov function associated to E_2

Proposition 5.1.6. *Let $t' \in F^\times$ with valuation $v(t') = -\mu$ for some integer $\mu > 0$. Take the character σ defined in* (5.1.1) *to be unramified. Furthermore, let ψ be an additive character with conductor $\mathfrak{p}^{-\mu}$.*

Take the Schwartz function ϕ to be

$$\phi = \mathrm{char}_{\mathcal{O}^\times \times (t' + \mathcal{O})} . \tag{5.1.28}$$

Then the associated Whittaker function $W_2 = W_2(g, s, \phi, \eta_1, \eta_2)$ is invariant *under $K_1(\mathfrak{p}^{\mu+1})$. Furthermore for $x \in F^\times$, we have*

$$W_2\begin{pmatrix} x & \\ & 1 \end{pmatrix} = \begin{cases} |x|^{1-s}(1-q^{-1})^{-1}, & x \in \mathcal{O}\backslash\{0\} \\ 0, & \textit{otherwise.} \end{cases} \tag{5.1.29}$$

Proof. Notice that since the chosen Schwartz function $\phi = \mathrm{char}_{\mathcal{O}^\times \times (t'+\mathcal{O})}$ is invariant under $K_1(\mathfrak{p}^{\mu+1})$, the associated section and hence the Whittaker function will also be right invariant under the same group.

By definition, we have for $x \in F^\times$:

$$\begin{aligned} W_2\begin{pmatrix} x & \\ & 1 \end{pmatrix} &= \int_F f_2\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right)\psi(u)du \\ &= |x|^{1-s}\sigma(-1)\int_{(t'+\mathcal{O})\times\mathcal{O}^\times} \sigma(t)\psi(uxt^{-1})dud^\times t \end{aligned} \tag{5.1.30}$$

Under our assumptions that σ is unramified and ψ has conductor $\mathfrak{p}^{-\mu}$, this integral is nonzero if $x \in \mathcal{O}\backslash\{0\}$, in which case we get:

$$W_2\begin{pmatrix} x & \\ & 1 \end{pmatrix} = |x|^{1-s}\int_{(t'+\mathcal{O})\times\mathcal{O}^\times} dud^\times t = |x|^{1-s}(1-q^{-1})^{-1}. \tag{5.1.31}$$

□

Proposition 5.1.7. *Let $\mathfrak{p} = (\varpi)$ be the maximal ideal of $\mathcal{O}$. Define the following function:*

$$\widetilde{W_2} = W_2 - \frac{1}{q}\sum_{x\in\mathbb{Z}/q\mathbb{Z}} \begin{pmatrix} 1 & \varpi^{-1}x \\ & 1 \end{pmatrix} W_2. \tag{5.1.32}$$

Then for all $h \in K_1(\mathfrak{p}^{\mu+1})$ and $m \in F^\times$, the Kirillov function $\widetilde{W_2}\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right)$ satisfies the following:

$$\widetilde{W_2}\left(\begin{pmatrix} m & \\ & 1 \end{pmatrix}h\right) = \mathrm{char}_{\mathcal{O}^\times}(m)W_2\begin{pmatrix} m & \\ & 1 \end{pmatrix}. \tag{5.1.33}$$

Proof. Since W_2 is right invariant under $K_1(\mathfrak{p}^{\mu+1})$, we can take $h = 1$.

As a result of Proposition 5.1.6, it suffices to consider $m \in \mathcal{O}\backslash\{0\}$.

Notice that since

$$\begin{pmatrix} m & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{x}{\varpi} \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{mx}{\varpi} \\ & 1 \end{pmatrix}\begin{pmatrix} m & \\ & 1 \end{pmatrix}, \tag{5.1.34}$$

we can apply the definition (5.1.32) with respect to the character ψ, that is:

$$\widetilde{W_2}\begin{pmatrix} m & \\ & 1 \end{pmatrix} = W_2\begin{pmatrix} m & \\ & 1 \end{pmatrix} - \frac{1}{q}\sum_{x\in\mathbb{Z}/q\mathbb{Z}} \psi\Big(\frac{mx}{\varpi}\Big) W_2\begin{pmatrix} m & \\ & 1 \end{pmatrix}. \tag{5.1.35}$$

Observe that if $m \in \mathcal{O}^\times$, the sum of $\psi(\frac{mx}{\varpi})$ is just the sum of all q-th roots of unity, thus 0. On the other hand, if $m \in \mathcal{O}\backslash\mathcal{O}^\times$, then $\frac{mx}{\varpi} \in \mathcal{O}$ and so $\psi(\frac{mx}{\varpi}) = 1$ for all $x \in \mathbb{Z}/q\mathbb{Z}$. This shows (5.1.33) holds.□

Computing the integral Now we abuse the notation and write W_2 to denote $\widetilde{W_2}$, the Whittaker function constructed in Proposition 5.1.7.

Recall that the Kirillov function $W\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right)$ is actually a Schwartz function on $F^\times$ (ref. [God70, p 1.24, Theorem 3]). Its support is compact and we assume that its intersection with $\mathcal{O}^\times$ is of the form

$$a + \mathfrak{p}^\nu \tag{5.1.36}$$

for some $\nu \geq 0$ and $a \in (\mathbb{Z}/q^\nu\mathbb{Z})^\times$. Here if $\nu = 0$, then $1 + \mathfrak{p}^\nu := \mathcal{O}^\times$.

Proposition 5.1.8. *Assume $W(1) = 1$. The following is valid at a bad prime:*

$$\mathcal{Z}(s_1, W_2, W, \rho) = q^{2-(\nu+\mu)}(q+1)^{-1}(q-1)^{-2}. \tag{5.1.37}$$

Proof. From (5.1.26), f_1 on $G(\mathcal{O})$ is only nonzero when it is restricted to $K_0(\mathfrak{p}^{\mu+1})$. Also we know from assumptions and constructions that W_2 and W are both $K_1(\mathfrak{p}^{\mu+1})$-invariant.

So from Proposition 5.1.7 the integral $\mathcal{Z}(s_1, W_2, W, \rho)$ in (5.1.6) is reduced to the following one:

$$\int_{K_0(\mathfrak{p}^{\mu+1})}\int_{\mathcal{O}^\times} f_1 W_2 W\left(\begin{pmatrix} x & \\ & 1\end{pmatrix} h\right) d^\times x dh. \tag{5.1.38}$$

Instead of integrating over K_0 we will do it over K_1. In fact, the above expression can be written as the following:

$$[K_0(\mathfrak{p}^{\mu+1}) : K_1(\mathfrak{p}^{\mu+1})]\int_{K_1(\mathfrak{p}^{\mu+1})}\int_{\mathcal{O}^\times} f_1 W_2 W\left(\begin{pmatrix} x & \\ & 1\end{pmatrix} h\right) d^\times x dh. \tag{5.1.39}$$

By construction, the integrand is invariant under K_1, so the above is equal to:

$$= \mathrm{Vol}(K_1(\mathfrak{p}^{\mu+1}))[K_0(\mathfrak{p}^{\mu+1}) : K_1(\mathfrak{p}^{\mu+1})]\int_{\mathcal{O}^\times} f_1 W_2 W\begin{pmatrix} x & \\ & 1\end{pmatrix} d^\times x. \tag{5.1.40}$$

As noted in (5.1.36), the integral in the above expression reduces to:

$$f_1(1)\int_{1+\mathfrak{p}^\nu} W_2\begin{pmatrix} x & \\ & 1\end{pmatrix} d^\times x. \tag{5.1.41}$$

From (5.1.27), we have

$$f_1(1) = 1. \tag{5.1.42}$$

By Proposition 5.1.6, we have for $x \in 1 + \mathfrak{p}^\nu$

$$W_2\begin{pmatrix} x & \\ & 1\end{pmatrix} = (1 - q^{-1})^{-1}. \tag{5.1.43}$$

Observe that

$$\mathrm{Vol}(K_1(\mathfrak{p}^{\mu+1}))[K_0(\mathfrak{p}^{\mu+1}) : K_1(\mathfrak{p}^{\mu+1})] = [G(\mathcal{O})\colon K_0(\mathfrak{p}^{\mu+1})]^{-1} = (1+q^{-1})^{-1}q^{-\mu-1}. \tag{5.1.44}$$

Now substituting results (5.1.42), (5.1.43) and (5.1.44) into (5.1.41), we get (5.1.37).□

5.1.3 Archimedean places

To be able to obtain explicit results, we assume that the infinite component φ_∞ of the cusp form φ satisfies the following condition:

$$\varphi_\infty \in \mathcal{B}(\chi_1\Theta_1, \chi_2\Theta_2). \tag{5.1.45}$$

We assume that W, W_1 and W_2 are constructed in the same way as in section 3.3.4.

At archimedean places, the role of the open compact group K at a nonarchimedean place is now played by $\mathrm{SO}_2(\mathbb{R})$ or $\mathrm{SU}_2(\mathbb{C})$, depending on whether the place is real or complex.

Real place

Proposition 5.1.9. *At a real place, the Rankin-Selberg integral has the following expression:*

$$\begin{aligned}&\int_{\mathrm{SO}_2(\mathbb{R})\times\mathbb{R}^\times} W_2 W f_1\left(\begin{pmatrix} t & \\ & 1\end{pmatrix} h\right) dh d^\times t\\ &= L_\mathbb{R}(2s_1+|k|)L_\mathbb{R}(2s_2+|l|)L_\mathbb{R}(2s_1+2s_2-1+|k+l|)J_\mathbb{R}(s_1,s_2,l,k).\end{aligned} \tag{5.1.46}$$

where

$$J_\mathbb{R}(s_1,s_2,l,k) := \int_{\mathbb{R}^\times} \rho(-t)|t|^{\frac{5}{2}-2s_2} I(s_2,l,t) I\Big(s_1+s_2-\frac{1}{2}, -l-k, t\Big) d^\times t \tag{5.1.47}$$

with I given by (3.3.45).

Proof. From Lemma 2.4.8 , we know

$$f_1\begin{pmatrix} t & \\ & 1\end{pmatrix} = \rho(t)|t|^{s_1} L_\mathbb{R}(2s_1+|k|). \tag{5.1.48}$$

From Proposition 3.3.9, we have

$$W_2 \begin{pmatrix} t & \\ & 1 \end{pmatrix} = \sigma(-1)|t|^{1-s_2} L_{\mathbb{R}}(2s_2 + |l|) I(s_2, l, t). \tag{5.1.49}$$

And from Proposition 3.3.9, we have

$$W \begin{pmatrix} t & \\ & 1 \end{pmatrix} = \overline{\sigma\rho}(-1)|t|^{\frac{3}{2}-(s_1+s_2)} L_{\mathbb{R}}(2s_1+2s_2-1+|k+l|) I\Big(s_1+s_2-\frac{1}{2}, -l-k, t\Big). \tag{5.1.50}$$

Now put them together and notice that the integrand is $\mathrm{SO}_2(\mathbb{R})$ invariant, we obtain that the Rankin-Selberg integral is:

$$\begin{aligned} &L_{\mathbb{R}}(2s_1 + |k|) L_{\mathbb{R}}(2s_2 + |l|) L_{\mathbb{R}}(2s_1 + 2s_2 - 1 + |k + l|) \cdot \\ &\int_{\mathbb{R}^\times} \rho(-t)|t|^{\frac{5}{2}-2s_2} I(s_2, l, t) I\Big(s_1 + s_2 - \frac{1}{2}, -l - k, t\Big) d^\times t. \end{aligned} \tag{5.1.51}$$

□

Using Lemma 3.3.10, we get the following:

Corollary 5.1.10. *If $l > 0$ and $k + l < 0$, then the Rankin-Selberg integral has the following simplified expression:*

$$4i^k L_{\mathbb{R}}(2s_1 + |k|) \tilde{J}_{\mathbb{R}}(s_1, s_2, l, k) \tag{5.1.52}$$

where

$$\begin{aligned} &\tilde{J}_{\mathbb{R}}(s_1, s_2, l, k) \\ &:= \int_{\mathbb{R}^\times} \rho(-t)|t|^{\frac{5}{2}-2s_2} \left(\sum_{m=0}^{l} \binom{l}{m} (-2\pi)^{-m} \cdot \frac{d^m}{dt^m} \Big[|t|^{s_2+\frac{l-1}{2}} K_{s_2+\frac{l-1}{2}}(2\pi|t|) \Big] \right) \cdot \\ &\left(\sum_{n=0}^{-l-k} \binom{-l-k}{n} (-2\pi)^{-n} \cdot \frac{d^n}{dt^n} \Big[|t|^{s_1+s_2-\frac{l+k}{2}-1} K_{s_1+s_2-\frac{l+k}{2}-1}(2\pi|t|) \Big] \right) d^\times t. \end{aligned} \tag{5.1.53}$$

Complex place

Proposition 5.1.11. *As in Lemma 2.4.10 and Proposition 3.3.11, we assume $\rho = |\ |^{2\lambda_1}, \sigma = |\ |^{2\lambda_2}$ for some purely imaginary numbers λ_1, λ_2. The following expression is valid at a complex place:*

$$\int_{\mathrm{SU}_2(\mathbb{C})\times\mathbb{C}^\times} W_2 W f_1\left(\begin{pmatrix} t & \\ & 1 \end{pmatrix} h\right) dh d^\times t = (2\pi)^2 L_\mathbb{C}(2s_1+\lambda_1) J_\mathbb{C}(s_1, s_2, \lambda_1, \lambda_2), \tag{5.1.54}$$

under the conditions:

$$\mathrm{Re}(s_2) > \frac{1}{8}, \quad \mathrm{Re}(s_1) > \frac{1}{2}. \tag{5.1.55}$$

Here

$$\begin{aligned} J_\mathbb{C}(s_1, s_2, \lambda_1, \lambda_2) = \int_{\mathbb{C}^\times} & |t|^{5-4s_2+2\lambda_1} (\mathrm{Re}\, t)^{2s_1+4s_2-\lambda_1-3} K_{2s_2+\lambda_2-1}(4\pi\, \mathrm{Re}\, t) \cdot \\ & K_{2s_1+2s_2-\lambda_1-\lambda_2-2}(4\pi\, \mathrm{Re}\, t) d^\times t. \end{aligned} \tag{5.1.56}$$

Proof. From formulae (2.4.10) and (3.3.53), we have

$$\begin{aligned} & f_1 \begin{pmatrix} t & \\ & 1 \end{pmatrix} = \rho(t)|t|^{2s_1} L_\mathbb{C}(2s_1+\lambda_1), \\ & W_2 \begin{pmatrix} t & \\ & 1 \end{pmatrix} = (2\pi)\sigma(-1)(\mathrm{Re}\, t)^{2s_2+\lambda_2-1}|t|^{2-2s_2} K_{2s_2+\lambda_2-1}(4\pi\, \mathrm{Re}\, t), \quad \mathrm{Re}(s_2) > \frac{1}{8}, \\ & \overline{W} \begin{pmatrix} t & \\ & 1 \end{pmatrix} = (2\pi)\overline{\sigma\rho}(-1)(\mathrm{Re}\, t)^{2s_1+2s_2-\lambda_1-\lambda_2-2}|t|^{3-2s_1-2s_2} K_{2s_1+2s_2-\lambda_1-\lambda_2-2}(4\pi\, \mathrm{Re}\, t), \\ & \mathrm{Re}(s_1+s_2) > \frac{5}{8}. \end{aligned} \tag{5.1.57}$$

Notice that each factor in the Rankin-Selberg integral is invariant under $\mathrm{SU}_2(\mathbb{C})$. Putting the above formulae together, we get the expression (5.1.54).
□

5.1.4 Global expressions

General case

We now put together local computations to get a global expression for the Rankin-Selberg integral.

Now F denotes a number field. Let S be the set of bad primes.

Putting together formulae (5.1.19), (5.1.37), (5.1.46) and (5.1.54), we get:

Theorem 5.1.12. *We have the following expression for the Rankin-Selberg integral:*

$$
\begin{aligned}
&(2\pi)^{-2r_2} \int_{G(F)Z(\mathbb{A})\backslash G(\mathbb{A})} \Omega dg \\
&= \prod_{v\ real} L_{\mathbb{R}}(2s_1 + |k_v|) L_{\mathbb{R}}(2s_2 + |l_v|) L_{\mathbb{R}}(2s_1 + 2s_2 - 1 + |k_v + l_v|) J_{\mathbb{R}}(s_1, s_2, l_v, k_v) \cdot \\
&\quad \prod_{v\ complex} L_{\mathbb{C}}(2s_1 + \lambda_{1,v}) J_{\mathbb{C}}(s_1, s_2, \lambda_{1,v}, \lambda_{2,v}) \prod_{v \in S} q_v^{2-(\nu_v+\mu_v)} (q_v+1)^{-1} (q_v-1)^{-2} \cdot \\
&\quad L_S(2s_1, \rho) L_S(2s_1+2, \rho)^{-1} L_S\left(\pi \otimes \bar{\omega}, s_1 + s_2 + \frac{1}{2}\right) L_S\left(\pi \otimes \rho, s_1 - s_2 + \frac{3}{2}\right).
\end{aligned}
\tag{5.1.58}
$$

Special case

Now let F be a totally real number field of degree d.

Definition 5.1.13. *By a holomorphic automorphic form on $G(\mathbb{A})$, we mean an automorphic form φ which is annihilated by the Maass lowering operators L_j for each $j, 1 \le j \le d$:*

$$L_j \varphi = 0, \tag{5.1.59}$$

where L_j is given in Iwasawa coordinates by:

$$L_j = e^{-2i\theta_j} \left\{ y_j \frac{\partial}{\partial x_j} + i y_j \frac{\partial}{\partial y_j} - \frac{1}{2} \frac{\partial}{\partial \theta_j} \right\}. \tag{5.1.60}$$

If φ is a holomorphic automorphic form on $G(\mathbb{A})$ with weight $k = (k_1, ..., k_d)$, $k_i \in \mathbb{Z}$ and central character ω which is trivial on the connected component of the identity in $Z(F_\infty)$, then the Whittaker function at an archimedean place is given by some exponential function. To be more precise, we know the following:

Proposition 5.1.14. *For $g \in G^+(\mathbb{R})$ of the following form:*

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y & \\ & 1 \end{pmatrix} \tag{5.1.61}$$

where $y \in \mathbb{A}^\times$ with every archimedean component being positive and $\epsilon \in F^\times$ which is totally positive, the following expression is valid of the Whittaker function W associated to φ:

$$W\left(\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix} g\right) = y^{\frac{k}{2}} \bar{\psi}(\epsilon x) e^{-2\pi\epsilon y}, \tag{5.1.62}$$

where k is the corresponding weight of φ at such a real place.

Proof. This is a reformulation of the first Proposition in [Gar90, p.102]. □

Taking product over all the archimedean places, we can write:

Corollary 5.1.15. *For g and ϵ given above,*

$$\prod_{v|\infty} W_v\left(\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix} g_v\right) = \bar{\Psi}(\epsilon x) e^{-2\pi \operatorname{Tr} \epsilon y} \prod_{v|\infty} y_v^{\frac{k_v}{2}}. \tag{5.1.63}$$

where $\Psi(\epsilon x) = \psi\left(\sum_{v|\infty} (\epsilon x)_v\right)$, and $\operatorname{Tr} \epsilon y = \sum_{v|\infty} (\epsilon y)_v$.

Now we go back to the problem of computing the Rankin-Selberg integral at an archimedean place.

Given a holomorphic Eisenstein series E_2 on $G(\mathbb{A})$ with weight l (multiple index, same below), central character σ and a holomorphic cusp form φ on $G(\mathbb{A})$ with weight $k + l$, central character ω. Take E_1 to be an Eisenstein series satisfying the following conditions:

(1) The central character ρ of E_1 satisfies $\rho\sigma\bar{\omega} = 1$.

(2) The weight of E_1 is k.

With the above conditions, the integrand in the Rankin-Selberg integral is invariant under $(\mathrm{SO}_2)^d$ and the centre $Z(F_\infty)$.

Notice that now the weight of φ is assumed to be $k+l$ as opposed to $-k-l$ assumed earlier. So the role of φ in the RS integral defined at the beginning of section 5.1 is now played by $\bar{\varphi}$ and the total weight in the integral is still zero.

Now at an archimedean place, the local RS integral has the following form:

Proposition 5.1.16. *For an archimedean place, suppose the weight of E_1, E_2 and φ are $k, l, k+l$ and $\rho(-1)(-1)^{|k|} = 1$. Then the following is valid:*

$$\mathcal{Z}(s_1, W_2, \overline{W}, \rho) = \begin{cases} 4^{-(s_1+l+\frac{k}{2})}\pi^{-(2s_1+l+k)}\Gamma(s_1+\frac{k}{2})\Gamma(s_1+l+\frac{k}{2}), & k \geq 0 \\ 4^{-(s_1+l+\frac{k}{2})}\pi^{-(2s_1+l)}\Gamma(s_1-\frac{k}{2})\Gamma(s_1+l+\frac{k}{2}), & k < 0 \end{cases} \tag{5.1.64}$$

Proof. Recall from Lemma 2.4.8 , we know for $t \in \mathbb{R}^\times$ and $\rho(-1)(-1)^{|k|} = 1$:

$$f_1\begin{pmatrix} t & \\ & 1 \end{pmatrix} = \rho(t)|t|^{s_1} L_{\mathbb{R}}(2s_1 + |k|). \tag{5.1.65}$$

Using Proposition 5.1.14, for $t > 0$:

$$\begin{aligned} W_2\begin{pmatrix} t & \\ & 1 \end{pmatrix} &= t^{\frac{l}{2}} e^{-2\pi t}, \\ W\begin{pmatrix} t & \\ & 1 \end{pmatrix} &= t^{\frac{k+l}{2}} e^{-2\pi t}. \end{aligned} \tag{5.1.66}$$

Now substituting the above expressions into the RS integral $\mathcal{Z}(s_1, W_2, \overline{W}, \rho)$ and noticing that we can take $t > 0$ in the integral since φ is holomorphic (ref. [Gar90, Remarks, p 104]), we get:

$$\begin{aligned}\mathcal{Z}(s_1, W_2, \overline{W}, \rho) &= \int_{\mathrm{SO}_2(\mathbb{R})\times\mathbb{R}_+^\times} W_2\overline{W} f_1\left(\begin{pmatrix} t & \\ & 1\end{pmatrix} h\right) dh d^\times t \\ &= L_{\mathbb{R}}(2s_1 + |k|)\int_0^\infty t^{s_1+l+\frac{k}{2}} e^{-4\pi t} d^\times t \\ &= L_{\mathbb{R}}(2s_1 + |k|)(4\pi)^{-(s_1+l+\frac{k}{2})}\Gamma(s_1 + l + \frac{k}{2}).\end{aligned} \tag{5.1.67}$$

Recalling expressions for $L_{\mathbb{R}}$ given in Lemma 2.4.8, we get the results.□

Good primes

At good primes, we need to modify the general result in (5.1.19) as the holomorphicity of E_2 entails a condition on s_2. In the special case of $F = \mathbb{Q}$, this condition is given by (iii) in Corollary 2.4.12. More generally,

Lemma 5.1.17. *If Eisenstein series E_2 given by* (2.4.14) *is holomorphic, then it is of parallel weight $(l, ..., l), l \in \mathbb{Z}$ and also $s_2 = \frac{l}{2}$.*

Proof. Since we know that the Maass lowering operator L_j is left-invariant, it suffices to consider L_j applied to the defining section f_2 at a real place, which by Lemma 2.4.8 is the following:

$$f_2(g) = L_{\mathbb{R}}(2s_2 + |l_j|)e^{il_j\theta_j}y_j^s, \quad g = \begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}}\end{pmatrix} h(\theta) \in \mathrm{SL}_2(\mathbb{R}). \tag{5.1.68}$$

Direct computation shows that $L_j f_2 = 0$ for all $j, 1 \le j \le d$ implies that l_j are all equal and $s_2 = \frac{l_j}{2} = \frac{l}{2}$.□

Now substituting $s_2 = \frac{l}{2}$ to the general expression (5.1.19) gives the following expression of the RS integral at a good prime:

Proposition 5.1.18.

$$\begin{aligned}\mathcal{Z}(s_1, \overline{W}, W_2, \rho) = &L(2s_1, \rho)L(2s_1 + 2, \rho)^{-1} L\left(\pi \otimes \omega, s_1 + \frac{l+1}{2}\right) \cdot \\ &L\left(\pi \otimes \rho, s_1 + \frac{3-l}{2}\right).\end{aligned} \tag{5.1.69}$$

The expressions at bad primes are the same as in the general expression (5.1.37).

Putting local expressions together

Proposition 5.1.19. *Let F be totally real of degree d. Let φ be a holomorphic cusp form with weight $(l+k_1,...,l+k_d)$ and central character ω which is trivial on the connected component of the identity in $Z(F_\infty)$. Let E_2 be a holomorphic Eisenstein series of weight $(l,...,l)$ and central character σ. Take E_1 to be an Eisenstein series of weight $(k_1,...,k_d)$ and central character ρ which satisfies $\rho\sigma\bar{\omega}=1$.*

At a bad prime in S, we choose the Schwartz function in f_1 to be $\phi = \mathrm{char}_{\mathfrak{p}^{\mu+1}\times\mathcal{O}^\times}$ as in Proposition 2.4.4 and that in E_2 to be $\phi = \mathrm{char}_{\mathcal{O}^\times\times(t'+\mathcal{O})}$ as in Proposition 5.1.6.

Take $\Omega = E_1E_2\bar{\varphi}$, then the Rankin-Selberg integral has the following form:

$$\begin{aligned}
&\int_{G(F)Z(\mathbb{A})\backslash G(\mathbb{A})} \Omega dg \\
&= \prod_{j=1}^{d} L_{\mathbb{R}}(2s_1+|k_j|)(4\pi)^{-(s_1+l+\frac{k_j}{2})}\Gamma(s_1+l+\frac{k_j}{2}) \prod_{v\in S} q_v^{2-(\nu_v+\mu_v)}(q_v+1)^{-1}(q_v-1)^{-2}\cdot \\
&\quad L_S(2s_1,\rho)L_S(2s_1+2,\rho)^{-1}L_S\left(\pi\otimes\omega, s_1+\frac{l+1}{2}\right)L_S\left(\pi\otimes\rho, s_1+\frac{3-l}{2}\right).
\end{aligned} \tag{5.1.70}$$

Bibliography

[Asa70] Tetsuya Asai. On a certain function analogous to $\log_\eta(z)$. *Nagoya Math. J.*, 40:193–211, 1970.

[Ati87] Michael Atiyah. The logarithm of the Dedekind η-function. *Math. Ann.*, 278(1-4):335–380, 1987.

[BJ79] A. Borel and H. Jacquet. Automorphic forms and automorphic representations. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 189–207. Amer. Math. Soc., Providence, R.I., 1979. With a supplement "On the notion of an automorphic representation" by R. P. Langlands.

[Bum97] Daniel Bump. *Automorphic forms and representations*, volume 55 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997.

[Cas73] W. Casselman. An assortment of results on representations of $\mathrm{GL}_2(k)$. In *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 1–54. Lecture Notes in Math., Vol. 349. Springer, Berlin, 1973.

[CdV83] Yves Colin de Verdière. Pseudo-laplaciens. II. *Ann. Inst. Fourier (Grenoble)*, 33(2):87–113, 1983.

[Cog04] James W. Cogdell. Lectures on L-functions, converse theorems, and functoriality for GL_n. In *Lectures on automorphic L-*

functions, volume 20 of *Fields Inst. Monogr.*, pages 1–96. Amer. Math. Soc., Providence, RI, 2004.

[Dri86] Yves Driencourt. Une présentation adélique de certaines fonctions analogues à $\log|\eta(z)|$. *J. Number Theory*, 23(1):111–135, 1986.

[EMOT53] Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi. *Higher transcendental functions. Vols. I, II.* McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. Based, in part, on notes left by Harry Bateman.

[Fla79] D. Flath. Decomposition of representations into tensor products. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 179–183. Amer. Math. Soc., Providence, R.I., 1979.

[Fri35] Kurt Friedrichs. Spektraltheorie halbbeschränkter Operatoren I. und II. Teil. *Math. Ann.*, 110(1):777–779, 1935.

[Gar90] Paul B. Garrett. *Holomorphic Hilbert modular forms.* The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1990.

[GJ79] Stephen Gelbart and Hervé Jacquet. Forms of GL(2) from the analytic point of view. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 213–251. Amer. Math. Soc., Providence, R.I., 1979.

[God64] Roger Godement. Analyse spectrale des fonctions modulaires. *Séminaire Bourbaki*, 9:15–40, 1964.

[God70] R. Godement. *Notes on Jacquet-Langlands theory.* Institute for Advanced Study, 1970.

[Gra07] I. S. (Izrail Solomonovich) Gradshtein. *Table of integrals, series, and products.* Elsevier Academic Press, Burlington, Mass. ; London, 7th ed edition, 2007.

[GS88] Stephen Gelbart and Freydoon Shahidi. *Analytic properties of automorphic L-functions*, volume 6 of *Perspectives in Mathematics.* Academic Press, Inc., Boston, MA, 1988.

[Hec18] E. Hecke. Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen. *Math. Z.*, 1(4):357–376, 1918.

[Jac72] Hervé Jacquet. *Automorphic forms on* GL(2)*. Part II.* Lecture Notes in Mathematics, Vol. 278. Springer-Verlag, Berlin-New York, 1972.

[JL70] H. Jacquet and R. P. Langlands. *Automorphic forms on* GL(2). Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin-New York, 1970.

[JL99] Jay Jorgenson and Serge Lang. Hilbert-Asai Eisenstein series, regularized products, and heat kernels. *Nagoya Math. J.*, 153:155–188, 1999.

[Kat04] Kazuya Kato. p-adic Hodge theory and values of zeta functions of modular forms. *Astérisque*, (295):ix, 117–290, 2004. Cohomologies p-adiques et applications arithmétiques. III.

[Kir95] *Representation theory and noncommutative harmonic analysis. II*, volume 59 of *Encyclopaedia of Mathematical Sciences.* Springer-Verlag, Berlin, 1995. Homogeneous spaces, representations and special functions, A translation of ıt Noncommutative harmonic analysis. 2 (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1990 [MR1099423 (91k:22002)], Translation edited by A. A. Kirillov.

[Kub73] Tomio Kubota. *Elementary theory of Eisenstein series.* Kodansha Ltd., Tokyo, 1973.

[Miy89] Toshitsune Miyake. *Modular forms.* Springer-Verlag, Berlin, 1989. Translated from the Japanese by Yoshitaka Maeda.

[Neu99] Jürgen Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[NS15] J Nekovár and A.J. Scholl. Introduction to plectic cohomology. 2015. *Preprint.*

[Ram64] K. Ramachandra. Some applications of Kronecker's limit formulas. *Ann. of Math. (2)*, 80:104–148, 1964.

[Rob73] Gilles Robert. *Unités elliptiques.* Société Mathématique de France, Paris, 1973. Bull. Soc. Math. France, Mém. No. 36, Bull. Soc. Math. France, Tome 101.

[RS72] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I. Functional analysis.* Academic Press, New York-London, 1972.

[Sch98] A. J. Scholl. An introduction to Kato's Euler systems. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 379–460. Cambridge Univ. Press, Cambridge, 1998.

[Sie65] Carl Ludwig Siegel. *Lectures on advanced analytic number theory.* Notes by S. Raghavan. Tata Institute of Fundamental Research Lectures on Mathematics, No. 23. Tata Institute of Fundamental Research, Bombay, 1965.

[Sta75] H. M. Stark. L-functions at $s = 1$. II. Artin L-functions with rational characters. *Advances in Math.*, 17(1):60–92, 1975.

[Tat67] J. T. Tate. Fourier analysis in number fields, and Hecke's zeta-functions. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 305–347. Thompson, Washington, D.C., 1967.

[WMLI92] M. Waldschmidt, P. Moussa, J. M. Luck, and C. Itzykson, editors. *From number theory to physics*. Springer-Verlag, Berlin, 1992. Papers from the Meeting on Number Theory and Physics held in Les Houches, March 7–16, 1989.

[WW96] E. T. Whittaker and G. N. Watson. *A course of modern analysis*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1996. An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions, Reprint of the fourth (1927) edition.

[Zag75] Don Zagier. A Kronecker limit formula for real quadratic fields. *Math. Ann.*, 213:153–184, 1975.